Welcome to the archival Web page for U.C. Berkeley's **Physics 105**, Section 2, Fall 2000. Email to: (Prof.) Mark Strovink, strovink@lbl.gov . I have a research web page, a standardized U.C. Berkeley web page, and a statement of research interests.

Most problem set solutions were first composed by Graduate Student Instructor Emory F. ("Ted") Bunn and later refined by GSI John Barber.
The GSI for this offering of the course was Robin Blume-Kohout.

Course documents:
Most documents linked here are in PDF format and are intended to be displayed by Adobe Acrobat [Reader], version 4 or later (Acrobat will do a better job if you **un**check "Use Greek Text Below:" on File-Preferences-General).

General Information including schedules and rooms.
Course Outline.
Lecture Notes (61 pp, 0.5 MB) used in Fall 1994, and from which readings are assigned in this course.

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Problem Set 1           Solution Set 1
Problem Set 2           Solution Set 2
Problem Set 3           Solution Set 3
Problem Set 4           Solution Set 4
Problem Set 5           Solution Set 5
Problem Set 6           Solution Set 6
Problem Set 7           Solution Set 7
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Problem Set 9           Solution Set 9
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Practice Midterm        Solution to Practice Midterm
Exam 1                  Solution to Exam 1
Exam 2                  Solution to Exam 2
Exam 3                  Solution to Exam 3
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Practice Final          Solution to Practice Final
Final Exam              Solution to Final Exam

Notes composed by GSI Robin Blume-Kohout:
P.S. #1 .pdf .ps
P.S. #2 .pdf .ps
P.S. #3 .pdf
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P.S. #4 .pdf
P.S. #5 .pdf
P.S. #6 .pdf
Hamilton's Principle

Mark Strovink

Professor
Particle Experiment

Mark Strovink, Ph.D. 1970 (Princeton). Joined UC Berkeley faculty in 1973 (Professor since 1980). Elected Fellow of the American Physical Society; served as program advisor for Fermilab (chair), SLAC (chair), Brookhaven, and the U.S. Department of Energy; served as D-Zero Physics Coordinator (1997 & 1998).

Research Interests

I am interested in experiments using elementary particles to test discrete symmetries, absolute predictions and other fundamental tenets of the Standard Model. Completed examples include early measurement of the parameters describing charge parity (*CP*) nonconservation in *K* meson decay; establishment of upper limits on the quark charge radius and early observation of the effects of gluon radiation in deep inelastic muon scattering; and establishment of stringent limits on right-handed charged currents both in muon decay and in proton-antiproton collisions, the latter via the search for production of right-handed *W* bosons in the D-Zero experiment at Fermilab.

After the discovery in 1995 by CDF and D-Zero of the top quark, we measured its mass with a combined 3% error, yielding (with other inputs) a stringent test of loop corrections to the Standard Model and an early hint that the Higgs boson is light. If a Higgs-like signal is seen, we will need to measure the top quark mass more than an order of magnitude better in order to determine whether that signal arises from the SM Higgs.

Current Projects

A continuing objective is to understand better how to measure the top quark mass. Top quarks are produced mostly in pairs; each decays primarily to $b + W$. The b's appear as jets of hadrons. Each W decays to a pair of jets or to a lepton and neutrino. For top mass measurement the most important channels are those in which either one or both of the W's decay into an electron or muon. For the single-lepton final states, we developed in 1994-96 and applied in 1997 a new technique that suppresses backgrounds (mostly from single W production) without biasing the apparent top mass spectra. For the dilepton final states, where backgrounds and systematic errors are lower but two final-state neutrinos are undetected rather than one, a likelihood *vs.* top mass must be calculated for each event. During 1993-96 we developed a new prescription for this calculation that averages over the (unmeasured) neutrino rapidities, and we used it in 1997 to measure the top mass to ~7% accuracy in this more sparsely populated channel. In both channels, further improvements to measurement technique as well as accumulation of larger samples will be necessary.

While studying data from the 1992-1996 CDF and D-Zero samples that contain both an electron and a muon, we became aware of three events that cannot easily be attributed either to top quark decay or to backgrounds. Generally this is because the transverse momenta of the leptons (electrons, muons, and neutrinos as inferred from transverse momentum imbalance) are unexpectedly large. We anticipate confirming data *e.g.* from the D-Zero run that began in 2001.

Transverse momentum imbalance is a broad signature for new physics. For example, in many supersymmetric models, *R*-parity conservation requires every superparticle to decay eventually to a lightest superparticle that, like the neutrino, can be observed only by measuring a transverse momentum imbalance. Reliable detection of this signature is one of the severest challenges for collider detectors. D-Zero's uniform and highly segmented uranium/liquid argon calorimeter yields the best performance achieved so far. Building on that, we have developed a new approach to analysis of transverse momentum imbalance that, for a given efficiency, yields up to five times fewer false positives.

Recently we have grappled with the long-standing problem of searching with statistical rigor for new physics in samples that should be describable by Standard Model processes – when the signatures for new physics are *not* strictly predefined. We have identified plausible methods for performing this type of analysis, and have exercised them on D-Zero data, but the methods involve sacrifices in sensitivity that we are still working to mitigate.

Selected Publications

S. Abachi *et al*. (D-Zero Collaboration), "Search for right-handed *W* bosons and heavy *W′* in proton-antiproton collisions at $\sqrt{s}$ = 1.8 TeV," *Phys. Rev. Lett*. **76**, 3271 (1996).

S. Abachi *et al*. (D-Zero Collaboration), "Observation of the top quark," *Phys. Rev. Lett*. **74**, 2422 (1995).

B. Abbott *et al*. (D-Zero Collaboration), "Direct measurement of the top quark mass," *Phys. Rev. Lett*. **79**, 1197 (1997); *Phys. Rev.* **D 58**, 052001 (1998).

B. Abbott *et al*. (D-Zero Collaboration), "Measurement of the top quark mass using dilepton events," *Phys. Rev. Lett*. **80**, 2063 (1998); *Phys. Rev.* **D 60**, 052001 (1999).

V.M. Abazov *et al*. (D-Zero Collaboration), "A quasi-model-independent search for new high p_T physics at D-Zero," *Phys. Rev. Lett*. **86**, 3712 (2001); *Phys. Rev.* **D 62**, 092004 (2000); *Phys. Rev.* **D 64**, 012004 (2001).

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University of California, Berkeley
Physics 105, Section 2, Fall 2000 (Strovink)

GENERAL INFORMATION (6 Sep 00)

Web site for this course: `http://d0lbln.lbl.gov/105f00-web.htm` . If this site is busy, as may occur during times of peak demand, use the partial mirror at `http://www.bestofberkeley.com` .

Instructors: Prof. **Mark Strovink**, 437 LeConte; (LBL) 486-7087; (home, before 10) 486-8079; (UC) 642-9685. Email: `strovink@lbl.gov` . Web: `http://d0lbln.lbl.gov` . Office hours: M 3:15-4:15, 5:30-6:30. Mr. **Robin Blume-Kohout**, 279 LeConte, (UC) 642-5647; (home, 10-10) 215-1541. Email: `rbk@socrates.berkeley.edu` . Office hours (in 279 LeConte): M Tu 2:30-3:30.

Lectures: TuTh 11:10-12:30 and W 5:10-6:30, all in 329 LeConte. The W 5:10-6:30 slot will be used at various times during the semester for the four 50-minute exams; the first two weeks of section; reviews and special lectures; and, occasionally, lectures to substitute for those that would normally be delivered on Th 11:10-12:30. Lecture attendance is strongly encouraged, since the course content is not exactly the same as that of the textbook. Sixty-one pages of typeset lecture notes used for Physics 105 in Fall 1994 are linked to the web site for the present course and are part of the **Reader**.

Discussion Sections: Tu 4:10-5 in 183 Dwinelle, and Th 5:10-6 in 343 LeConte. During the first two weeks, because of room unavailability, there will be only one combined section: W 5:10-6 in 329 LeConte. Taught by Mr. Blume-Kohout. You are especially encouraged to attend discussion section regularly. There you will learn techniques of problem solving, with particular application to the assigned exercises.

Text (required): Hand and Finch, **Analytical Mechanics** (Cambridge paperback or hardcover, 1998).

Reader (required): 88 pp of typeset material. Print it from the web, or buy it at Copy Central for ~$6.

Problem Sets: A required and most important part of the course. Thirteen problem sets are assigned and graded. For each assigned problem, the grading scale is the same: “0” = “problem was not attempted”; “1” = “problem was attempted”. If your solution is incorrect, the grader will endeavor to write comments that will help you see where you went astray. Solutions to *all* assigned problems *are now available* on the course web site. Problem sets are due on Mondays at 4:15 PM, beginning in week 2, except for Thanksgiving week. Deposit problem sets in the box labeled “105 Section 2 (Strovink)” in the second floor breezeway between LeConte and Birge Hall. You are encouraged to attempt all of the problems. Students who do not do so find it almost impossible to learn the material and to succeed on the examinations. Late papers will not be graded. Your lowest problem set score will be dropped, in lieu of due date extensions for any reason.

Exams: There will be four 50-minute examinations (one exam every three weeks, beginning in the third week), and one 3-hour final examination. Before confirming your enrollment in this class, please check that its final Exam Group 9 does not conflict with the Exam Group for any other class in which you are enrolled. Please verify now that you will be available for all four of the 50-minute examinations, on W 13 Sep, W 4 Oct, W 25 Oct, and W 15 Nov, 5:10-6:00 PM; and for the final examination, F 15 Dec, 5-8 PM. Your lowest 50-minute exam score will be dropped, in lieu of absences for any reason. Except for unforeseeable emergencies, it will not be possible for the final exam to be rescheduled. Passing 105 requires passing the final exam.

Grading: 20% problem sets, 40% 50-minute examinations, 40% final exam. Departmental regulations call for an *A*:*B*:*C* distribution in the ratio 2:3:2, with approximately 10-15% *D*’s or *F*’s. However, the fraction of *D*’s or *F*’s depends on you; no minimum number need be given.

COURSE OUTLINE

Week No.	Week of..	Topic	Problem Set No.	Due at 4:15 PM on..
1	28-Aug	Vectors and transformations.		
	4-Sep	LABOR DAY HOLIDAY		
2	4-Sep	Selective review of Newtonian mechanics. Calculus of variations.	1	5-Sep
3	11-Sep	Calculus of variations (cont'd).	2	11-Sep
	13-Sep	50-minute **Exam 1**		
4	18-Sep	Lagrangian mechanics.	3	18-Sep
5	25-Sep	Lagrangian mechanics (cont'd). Linear oscillators.	4	25-Sep
6	2-Oct	Linear oscillators (cont'd).	5	2-Oct
	4-Oct	50-minute **Exam 2**		
7	9-Oct	Central forces.	6	9-Oct
8	16-Oct	Central forces (cont'd). Scattering.	7	16-Oct
9	23-Oct	Hamiltonian dynamics.		
	25-Oct	50-minute **Exam 3**		
10	30-Oct	Rotational motion.	8	30-Oct
11	6-Nov	Rotational motion (cont'd). Coupled oscillations.	9	6-Nov
12	13-Nov	Coupled oscillations (cont'd). Waves.	10	13-Nov
	15-Nov	50-minute **Exam 4**		
13	20-Nov	Waves (cont'd).	11	20-Nov
	23-Nov	THANKSGIVING HOLIDAY		
14	27-Nov	Nonanalytic problems.	12	27-Nov
15	4-Dec	Chaos.	13	4-Dec
	7-Dec	LAST LECTURE (review).		
16	11-Dec			
	13-Dec	FINAL EXAMINATIONS begin		
	15-Dec	180-minute **Final Examination,** F 15 Dec 5-8 PM		

SHORT COURSE IN CLASSICAL MECHANICS

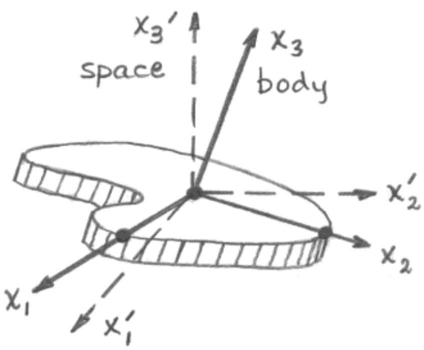
M. Strovink
University of California, Berkeley

January 2, 2001

1. Vectors and Transformations.

1.1. Body and space axes.

In ordinary 3-dimensional space, we require six independent quantities to specify the configuration of a rigid body. (Take $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ to be vectors from the origin to each of three reference points in the body. If the body is rigid, $|\mathbf{r}_1 - \mathbf{r}_2|$, $|\mathbf{r}_2 - \mathbf{r}_3|$, and $|\mathbf{r}_3 - \mathbf{r}_1|$ are fixed, so the number of independent quantities is only six.) Three of these six may be identified with a vector $\mathbf{R}$ from the origin to some basic reference point, e.g. the center of mass. The remaining three quantities are *orientation variables*.



To study these orientation variables, we use a set of unprimed "body axes" (x_1, x_2, x_3) attached to the rigid body, and a set of primed "space axes" (x_1', x_2', x_3') using the same origin as the body axes, but having their directions fixed to be the same as those of a particular set of external reference axes. The external frame is usually taken to be inertial, i.e. not accelerating. A point on the rigid body can be represented in either coordinate system.

Going from one representation to another requires a *linear transformation*:

$$\begin{aligned} x_1' &= \lambda_{11}x_1 + \lambda_{12}x_2 + \lambda_{13}x_3 \\ x_2' &= \lambda_{21}x_1 + \lambda_{22}x_2 + \lambda_{23}x_3 \\ x_3' &= \lambda_{31}x_1 + \lambda_{32}x_2 + \lambda_{33}x_3 \end{aligned} \tag{1.1}$$

(We require the transformation to be linear so that it does not depend on the dimensions of x and x'.) Other notation for (1.1) is:

$$x_i' = \sum_{j=1}^{3} \lambda_{ij} x_j$$

$$x_i' = \lambda_{ij} x_j$$

where in the last expression summation from 1 to 3 over the *repeated index* j is *assumed by convention*. In matrix notation, we could also write

$$\tilde{x}' = \Lambda \tilde{x}$$

where

$$\tilde{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \tilde{x}' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$$

are column vectors, and

$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix}$$

is a 3×3 matrix. Of course, the rules of matrix multiplication are followed.

To test your understanding of matrix multiplication and the convention that repeated indices are summed, consider the product

$$C = AB$$

where A, B, and C are 3×3 matrices. Then the ij element of C is given by

$$C_{ij} = A_{ik} B_{kj} .$$

1.2. Properties of the transformation matrix.

The nine matrix elements $\lambda_{11} \ldots \lambda_{33}$ depend on only three (as yet unspecified) orientation variables. Therefore, there must be six equations that relate the matrix elements to each other (more on that below). The physical significance of the λ_{ij} is revealed by transforming the *unit vectors*

$$\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \hat{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

into the primed system, as usual by multiplying them by Λ:

$$\Lambda \hat{e}_1 = \begin{pmatrix} \lambda_{11} \\ \lambda_{21} \\ \lambda_{31} \end{pmatrix}.$$

Therefore, λ_{11} is the projection of $\hat{e}_1$ on the x'_1 axis, λ_{21} is the projection of $\hat{e}_1$ on the x'_2 axis, etc.

Using the well-known property of the dot product

$$\cos\theta_{ab} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|},$$

we obtain

$$\begin{aligned} \lambda_{11} &= \hat{e}'_1 \cdot \hat{e}_1 = \cos\theta_{1'1} \\ \lambda_{21} &= \hat{e}'_2 \cdot \hat{e}_1 = \cos\theta_{2'1} \\ \lambda_{31} &= \hat{e}'_3 \cdot \hat{e}_1 = \cos\theta_{3'1}, \end{aligned}$$

etc. That is, the λ_{ij} are the *direction cosines* relating axis i' to axis j.

We return to the six equations relating the λ_{ij} to each other. These may be obtained from trigonometry, or more instructively by considering the need to *preserve the dot product* under transformation. In matrix notation,

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_i b_i.$$

It is useful to express the dot product in terms of the *transpose* of a matrix. We denote the transpose of A by A^t, following the definition

$$A^t_{ij} \equiv A_{ji},$$

signifying the interchange of rows and columns. Then the dot product becomes

$$\mathbf{a} \cdot \mathbf{b} = \tilde{a}^t \tilde{b},$$

where, as usual, $\tilde{a}$ and $\tilde{b}$ are the column vectors.

Now we require that $\tilde{a}^t \tilde{b}$ remain invariant in the primed system, related to the unprimed system by the transformation Λ:

$$\tilde{a}' = \Lambda \tilde{a}; \quad \tilde{b}' = \Lambda \tilde{b}.$$

We want

$$\tilde{a}'^t \tilde{b}' = \tilde{a}^t \tilde{b}.$$

To proceed further, we need to know how to take the transpose of a product of matrices. In general, for any two matrices A and B,

$$[(AB)^t]_{ij} = [AB]_{ji} = A_{jk} B_{ki} = (A^t)_{kj} (B^t)_{ik}$$

$$= (B^t)_{ik} (A^t)_{kj} = (B^t A^t)_{ij}.$$

That is, the transpose of a product of matrices is the product of the transposed matrices multiplied in the opposite order:

$$(AB)^t = B^t A^t.$$

Returning to the invariance of the dot product, we have

$$\tilde{a}'^t \tilde{b}' = (\Lambda \tilde{a})^t \Lambda \tilde{b} = \tilde{a}^t \Lambda^t \Lambda \tilde{b}.$$

This can be equal to $\tilde{a}^t \tilde{b}$ for *all possible* $\tilde{a}$ and $\tilde{b}$ if and only if the product $\Lambda^t \Lambda$ reduces to the unit matrix:

$$\Lambda^t \Lambda = \mathrm{I}.$$

If this condition is satisfied, Λ is said to be *orthogonal*. Expressed in component form, the orthogonality requirement is

$$\lambda^t_{ik} \lambda_{kj} = \lambda_{ki} \lambda_{kj} = \delta_{ij},$$

where the *Kronecker delta* δ_{ij} is 1 if $i = j$, 0 otherwise.

Since i and j each can range through three values, this last equation is really nine equations. They are:

$$\begin{aligned}
\lambda_{11}\lambda_{11} + \lambda_{21}\lambda_{21} + \lambda_{31}\lambda_{31} &= 1 \\
\lambda_{12}\lambda_{12} + \lambda_{22}\lambda_{22} + \lambda_{32}\lambda_{32} &= 1 \\
\lambda_{13}\lambda_{13} + \lambda_{23}\lambda_{23} + \lambda_{33}\lambda_{33} &= 1 \\
\lambda_{11}\lambda_{12} + \lambda_{21}\lambda_{22} + \lambda_{31}\lambda_{32} &= 0 \\
\lambda_{12}\lambda_{11} + \lambda_{22}\lambda_{21} + \lambda_{32}\lambda_{31} &= 0 \\
\lambda_{11}\lambda_{13} + \lambda_{21}\lambda_{23} + \lambda_{31}\lambda_{33} &= 0 \\
\lambda_{13}\lambda_{11} + \lambda_{23}\lambda_{21} + \lambda_{33}\lambda_{31} &= 0 \\
\lambda_{12}\lambda_{13} + \lambda_{22}\lambda_{23} + \lambda_{32}\lambda_{33} &= 0 \\
\lambda_{13}\lambda_{12} + \lambda_{23}\lambda_{22} + \lambda_{33}\lambda_{32} &= 0
\end{aligned}$$

As we expected, these nine equations are really only six, because the last three pairs are identical.

The following complex generalizations are useful in quantum mechanics and elsewhere:

$$\begin{gathered}
\text{transpose } (A^t)_{ij} \equiv A_{ji} \to \text{adjoint } (A^\dagger)_{ij} \equiv A^*_{ji}; \\
\text{orthogonality } A^t A = \mathrm{I} \to \text{unitarity } A^\dagger A = \mathrm{I}; \\
\text{symmetric } A^t = A \to \text{Hermitian } A^\dagger = A.
\end{gathered}$$

1.3. Parity inversion.

Now we consider the determinant of Λ. In order to express the determinant in component form, we need the *Levi-Civita density* ϵ_{ijk}. This object is most straightforwardly used in the cross product:

$$\mathbf{a} \times \mathbf{b} \equiv \epsilon_{ijk} \hat{e}_i a_j b_k. \tag{1.2}$$

Given the rules for cross products, it must be true that

$$\begin{aligned}
\epsilon_{ijk} &= 0 \text{ unless } i \neq j \neq k \\
&= 1 \text{ for } i, j, k = \text{cyclic permutation of } 1, 2, 3 \\
&= -1 \text{ for } i, j, k = \text{cyclic permutation of } 3, 2, 1.
\end{aligned}$$

Using the Levi-Civita density, we can write the determinant of a 3×3 matrix as

$$\det A \equiv |A| \equiv \frac{1}{3!} \epsilon_{ijk} A_{il} A_{jm} A_{kn} \epsilon_{lmn}. \tag{1.3}$$

As i, j, k, l, m, n each run from 1 to 3, there are a total of $3^6 = 729$ terms. However, because the Levi-Civita density is usually zero, only 36 terms survive. Each is the product of three matrix elements that come from rows and columns which must be different. Each of these products, for example $A_{11}A_{22}A_{33}$, comes in six permutations, e.g. 123, 231, 312, 321, 132, 213. That's the reason for the 1/3! factor – only six of the 36 terms are not duplicates. Thus we confirm that this expression for the determinant gives the same result as does the standard "diagonal lines" mnemonic.

Writing the determinant using the Levi-Civita density enables us to conclude immediately that, for any 3×3 matrix,

$$\det A = \det A^t. \tag{1.4}$$

(In (1.3), changing A to A^t just interchanges i, j, k with l, m, n, yielding the same result.) It is also true that

$$\det AB = \det A \det B. \tag{1.5}$$

Equations (1.4) and (1.5) are true for (square) matrices of any size.

Returning to the rotation matrix Λ,

$$\begin{aligned}
1 &= \det \mathrm{I} = \det \Lambda^t \Lambda = (\det \Lambda)^2 \\
\pm 1 &= \det \Lambda.
\end{aligned}$$

An *infinitesimal rotation*,

$$\Lambda = \mathrm{I} + \mathcal{E}; \quad \det \mathcal{E} \ll 1$$

must be of the class

$$\det \Lambda = +1$$

since $\det \mathrm{I} = +1$. Any *finite* rotation can be built up from infinitesimal rotations and so must also have $\det \Lambda = +1$. If instead we have a transformation with $\det \Lambda = -1$, it must be related through a set of infinitesimal rotations not to I but to the *parity inversion* matrix

$$P \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \det P = -1.$$

1.4. Infinitesimal rotations.

Having found that the rotation matrix Λ is specified by three independent quantities, we arrive at a philosophical problem: can Λ be represented by a (3-component) vector, instead of a matrix, or tensor? For example, its direction could be the rotation axis, and its magnitude could be the angle of rotation. The answer turns out to be yes, but only if the rotation is infinitesimal.

To pursue this question further, we are reminded that, formally, a vector is any 3-component object whose components transform under rotations according to Λ. Vector addition is *commutative* and *associative*; multiplication by a *scalar* (invariant to rotations) is *commutative* and *distributive*. The dot product is *commutative*; and, as we see in (1.2), the cross product is *anticommutative*,

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a},$$

due to the change of sign of ϵ_{ijk} when any two indices are switched.

If the rotation could be represented by a vector, two successive rotations $\vec{\lambda}_1$ and $\vec{\lambda}_2$ would be described by another vector which is their sum $\vec{\lambda}_1 + \vec{\lambda}_2$. However, since vector addition is commutative, $\vec{\lambda}_1 + \vec{\lambda}_2 = \vec{\lambda}_2 + \vec{\lambda}_1$, the order of the rotations wouldn't matter. But physically it does! (Try rotating a book by 90° first through a vertical axis, then through a horizontal axis; and then vice versa.) On the other hand, if the rotation were represented by a tensor, as in general it is, we would ask that $\Lambda_1\Lambda_2 \neq \Lambda_2\Lambda_1$, as is generally the case, so that the order of rotations *would* matter.

Infinitesmal rotations *can* be described by a vector. Suppose

$$\begin{aligned} \Lambda_1 &= \mathrm{I} + \mathcal{E}_1, \quad \det \mathcal{E}_1 \ll 1 \\ \Lambda_2 &= \mathrm{I} + \mathcal{E}_2, \quad \det \mathcal{E}_2 \ll 1. \end{aligned}$$

Then the order of rotations doesn't matter:

$$\begin{aligned} \Lambda_1\Lambda_2 &= (\mathrm{I} + \mathcal{E}_1)(\mathrm{I} + \mathcal{E}_2) \\ &= \mathrm{I} + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_1\mathcal{E}_2 \\ &\approx \mathrm{I} + \mathcal{E}_1 + \mathcal{E}_2 \text{ (last term negligible)} \\ &= \Lambda_2\Lambda_1 \end{aligned}$$

Requiring Λ to be orthogonal demands that $\mathcal{E}$ be antisymmetric:

$$\begin{aligned} \mathrm{I} &= \Lambda^t\Lambda \\ &= (\mathrm{I} + \mathcal{E}^t)(\mathrm{I} + \mathcal{E}) \\ &= \mathrm{I} + \mathcal{E}^t + \mathcal{E} + \mathcal{E}^t\mathcal{E} \\ &\approx \mathrm{I} + \mathcal{E}^t + \mathcal{E} \\ \mathcal{E}^t &= -\mathcal{E}. \end{aligned}$$

An antisymmetric matrix has diagonal elements equal to zero, with only three independent off-diagonal elements. We choose the following general form for the infinitesimal rotation:

$$\mathcal{E} \equiv \begin{pmatrix} 0 & -d\Omega_3 & +d\Omega_2 \\ +d\Omega_3 & 0 & -d\Omega_1 \\ -d\Omega_2 & +d\Omega_1 & 0 \end{pmatrix},$$

where the d's emphasize smallness.

With $\mathcal{E}$ in this form, application to the vector $\mathbf{r}$ of the infinitesimal rotation produces an elegant formula:

$$\begin{aligned} \tilde{r}' &= \Lambda\tilde{r} \\ &= (\mathrm{I} + \mathcal{E})\tilde{r} \\ &= \tilde{r} + \mathcal{E}\tilde{r} \\ \tilde{r}' - \tilde{r} &= \mathcal{E}\tilde{r} \\ &= \begin{pmatrix} d\Omega_2 x_3 - d\Omega_3 x_2 \\ d\Omega_3 x_1 - d\Omega_1 x_3 \\ d\Omega_1 x_2 - d\Omega_2 x_1 \end{pmatrix} \end{aligned}$$

Defining $\tilde{r}' - \tilde{r} \equiv d\tilde{r}'$, this becomes

$$d\mathbf{r}' = d\vec{\Omega} \times \mathbf{r}. \tag{1.6}$$

In (1.6) $d\vec{\Omega}$ is a "vector" with components defined by the matrix elements of $\mathcal{E}$. Its direction is the axis about which the body (unprimed) axes have been rotated relative to the space axes. Equation (1.6) describes the difference $d\mathbf{r}'$ between the description $\mathbf{r}'$ of a space point as seen in the space axes and the description $\mathbf{r}$ of the same space point as seen in the (rotated) body axes. If the space point is *at rest* in the body system, all its motion as seen in the space system will be due to the body rotation and will be "tangential", or perpendicular to $\mathbf{r}$:

$$\begin{aligned}\mathbf{v}'_{\text{tang}} \equiv \frac{d\mathbf{r}'}{dt} &= \frac{d\vec{\Omega}}{dt} \times \mathbf{r} \\ &\equiv \vec{\omega} \times \mathbf{r},\end{aligned} \tag{1.7}$$

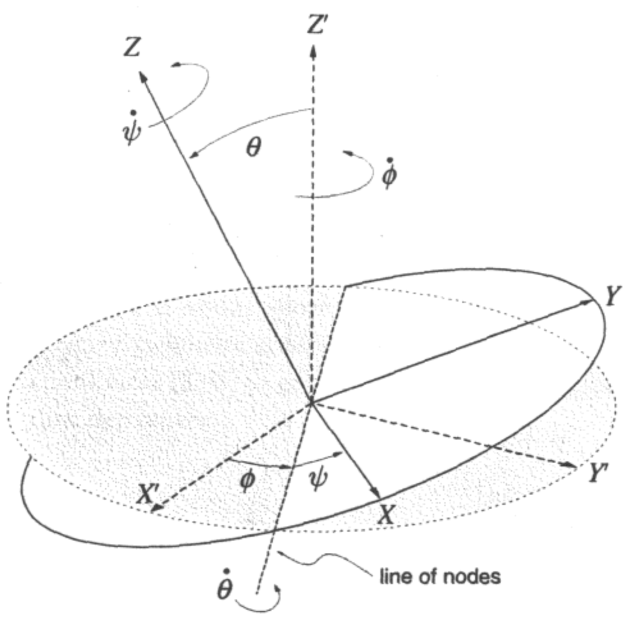
where $\vec{\omega}$ is the *angular velocity*. More generally, the space point will be moving with respect to the body system. Then its velocity in the space system will be the sum of the velocity in the body system and the velocity due to rotation:

$$\begin{aligned}\mathbf{v}' &= \mathbf{v} + \vec{\omega} \times \mathbf{r} \\ \frac{d\mathbf{P}'}{dt} &= \frac{d\mathbf{P}}{dt} + \vec{\omega} \times \mathbf{P}\end{aligned} \tag{1.8}$$

In the last equation, $\mathbf{v}'$ has been replaced by $d\mathbf{P}'/dt$ to emphasize that this transformation rule is valid for any vector $\mathbf{P}'$, not just the position vector.

We can prove that $\vec{\omega}$ is a vector only by examining in detail its transformation properties under coordinate rotation. It turns out that $\vec{\omega}$ *is* a type of vector ("axial vector" or "pseudovector") which is like an ordinary vector ("polar vector") exept that it does *not* change sign under parity inversion. Another familiar example of an axial vector is the magnetic field $\mathbf{B}$.

1.5. Euler rotation.



As an example of a 3-dimensional rotation, we introduce the *Eulerian angles*. The Euler rotation is *passive* (axes are rotated, not objects) and, by convention, it transforms from the space axes to the body axes. This is the *inverse* of the transformation we have been considering, so we denote the Euler rotation matrix by Λ^t rather than Λ:

$$\tilde{x} = \Lambda^t \tilde{x}'.$$

The Euler rotation is useful because it simplifies the analysis of certain problems such as tops, and because it is an established convention.

The Euler rotation consists of three steps. The first transformation, between the $(')$ and $('')$ frames, is a counterclockwise ("CCW") rotation about the $3'$ axis by the first Euler angle ϕ. The second transformation, between the $('')$ and $(''')$ frames, is a CCW rotation about the $1''$ axis (the "line of nodes") by the second Euler angle θ. The final step, a transformation between the $(''')$ and unprimed (body) frames, is a CCW rotation about the $3'''$ axis by the third Euler angle ψ. Note that, if θ were zero, the ϕ and ψ rotations would occur about the same axis and could be combined into a single rotation.

Expressed in rotation matrices,

$$\begin{aligned}\tilde{x}'' &= \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{x}' \\ &\equiv \Lambda^t_\phi \tilde{x}' \\ \tilde{x}''' &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tilde{x}'' \\ &\equiv \Lambda^t_\theta \tilde{x}'' \\ \tilde{x} &= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{x}''' \\ &\equiv \Lambda^t_\psi \tilde{x}'''\end{aligned} \tag{1.9}$$

Within each of these three matrices, the nontrivial 2×2 submatrices are identical to the usual two-dimensional rotation matrices that are used for more elementary transformations.

To obtain the full rotation matrix, one applies the individual rotations in order:

$$\tilde{x} = \Lambda^t_\psi \Lambda^t_\theta \Lambda^t_\phi \tilde{x}' = \Lambda^t \tilde{x}'.$$

Note that the first rotation, Λ^t_ϕ, is the *right-hand* factor in the product of matrices. The elements of the full rotation matrix, obtained by carrying out the matrix multiplication, are:

$$\Lambda^t = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} \tag{1.10}$$

2. Selective Review of Newtonian Mechanics.

2.1. Definitions.

Newton's laws are the consequence of the definition of force

$$\mathbf{F} \equiv \frac{d\mathbf{p}}{dt} \equiv \frac{d}{dt} m\mathbf{v} \tag{2.1}$$

in which the inertial mass m is the "resistance to change in velocity" and is proportional to the gravitational coupling constant, or gravitational mass. By convention, this constant of proportionality is unity. The fact that the two types of mass are proportional to each other is a consequence of general relativity and has been tested by balancing materials of high and low atomic number Z (in high Z materials, a larger fraction of the mass is due to relativistic effects that depend on the inertial mass). This proportionality has been verified to better than one part in 10^{14}. Equation (2.1) can be integrated to solve routine problems involving ballistics, rockets, etc.

The following analogies between linear and angular motion are important:

Quantity	**Linear motion**	**Angular motion**
Coordinate	$\mathbf{r}$	θ
Derivative of coordinate	$\mathbf{v} \equiv d\mathbf{r}/dt$	$\vec{\omega} \equiv \hat{e} d\theta/dt$ ($\hat{e}$ along axis of CCW rotation)
Momentum	$\mathbf{p} \equiv m\mathbf{v}$	$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}$
Derivative of momentum	$\mathbf{F} \equiv d\mathbf{p}/dt$	$\mathbf{N} \equiv d\mathbf{L}/dt = \mathbf{r} \times \mathbf{F}$

Missing in the above table is the relationship between $\mathbf{L}$ and $\vec{\omega}$. For a sufficiently simple rigid body (details later),

$$\mathbf{L} = I\vec{\omega}$$

where the moment of inertia I is a scalar. More generally, $\mathbf{L}$ is not parallel to $\vec{\omega}$, so I must be a tensor, the "inertia tensor", represented by a symmetric 3×3 matrix.

2.2. Relations between $\mathbf{r}$, $\mathbf{v}$, and $\vec{\omega}$ for a point particle.

When the body is a *point*, no orientation is defined, and only three coordinates are needed to specify its position. We do not need body axes, so, for the time being, we drop the primes from the space axes.

Without the primes, Eq. (1.8) becomes

$$\mathbf{v}_{\text{tang}} = \vec{\omega} \times \mathbf{r}.$$

Taking the cross product of $\mathbf{r}$ with it,

$$\mathbf{r} \times \mathbf{v}_{\text{tang}} = \mathbf{r} \times \mathbf{v} = \mathbf{r} \times (\vec{\omega} \times \mathbf{r}).$$

Using the "*bac cab*" rule,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}),$$

$$\mathbf{r} \times \mathbf{v} = \vec{\omega}(\mathbf{r} \cdot \mathbf{r}) - \mathbf{r}(\mathbf{r} \cdot \vec{\omega}).$$

The last term is that part of $r^2\vec{\omega}$ which is parallel to $\mathbf{r}$. After this parallel part is subtracted, what remains is the perpendicular part:

$$\begin{aligned} \mathbf{r} \times \mathbf{v} &= r^2 \vec{\omega}_{\perp \text{ to } \mathbf{r}} \\ \mathbf{L} &= m r^2 \vec{\omega}_{\perp \text{ to } \mathbf{r}}. \end{aligned}$$

Even for a point particle, $\mathbf{L}$ is equal to $mr^2\vec{\omega}$ only if $\vec{\omega}$ is $\perp$ to $\mathbf{r}$.

2.3. Work and energy.

The *work* done on a particle on a path between points 1 and 2 is

$$W_{12} \equiv \int_1^2 \mathbf{F} \cdot d\mathbf{r} \equiv T_2 - T_1,$$

where T is the *kinetic energy* $\frac{1}{2}mv^2$ and the integral is taken over the path. The vector force field $\mathbf{F}$ is called *conservative* if it can obtained from the gradient of a scalar field. The scalar field is conventionally written as $-U$ where U is the *potential energy*. If the force is conservative,

$$\begin{aligned} \mathbf{F} &= -\nabla U \\ W_{12} &= \int_1^2 -\nabla U \cdot d\mathbf{r} \\ &= -(U_2 - U_1) \\ T_2 - T_1 &= U_1 - U_2 \\ T_2 + U_2 &= T_1 + U_1, \end{aligned} \tag{2.2}$$

and total energy $T + U$ is conserved.

In the above situation in which $\mathbf{F}$ is conservative, W_{12} is path-independent. More generally, what conditions does path-independence of W_{12} place on $\mathbf{F}$? We consider any path from 1 to 2, combined with any other path from 2 back to 1. Since W_{12} is path-independent, and $W_{21} = -W_{12}$, it follows directly that the circuital integral of $\mathbf{F}$ around the combined path must vanish:

$$\oint \mathbf{F} \cdot d\mathbf{r} = W_{12} + W_{21} = 0.$$

What conditions on $\mathbf{F}$ are imposed?

First, $\mathbf{F}$ must be velocity-independent. Otherwise, we could negotiate the path 12 rapidly, and 21 slowly, spoiling the cancellation. Second, $\mathbf{F}$ must not depend explicitly on the time (as opposed to implicitly, for example as a result of particle motion). Otherwise, we could negotiate paths 12 and 21 at different times, again spoiling the cancellation. Finally, assuming that $\mathbf{F}$ is both velocity- and time-independent, it must be free of *circulation* in order that the circuital integral vanish. Qualitatively, a vector field with circulation loops back on itself. To get a more quantitative condition, we need Stokes' theorem:

$$\oint \mathbf{F} \cdot d\mathbf{r} = \iint (\nabla \times \mathbf{F}) \cdot d\mathbf{A},$$

where the surface integral applies to any surface bounded by the circuital path, and the element of area $d\mathbf{A}$ points outward according to the right-hand rule when applied to the loop. For the right-hand integral to be zero for any path and thus over any area, the integrand must vanish:

$$\nabla \times \mathbf{F} = 0.$$

The most general form of $\mathbf{F}$ with vanishing curl is:

$$\mathbf{F} = -\nabla U$$

because, formally,

$$\nabla \times (\nabla U) = 0.$$

Thus we are back to the original definition of a conservative force.

A qualitative evaluation of whether a vector field has circulation may not be completely reliable. Consider the magnetic field outside a long straight thin current-carrying wire oriented along $\hat{z}$. The field points in the θ direction and so obviously loops back along itself. However, since this field varies as $1/r$, where r is the perpendicular distance to the wire, it has no curl and therefore no circulation away from the singularity at $r = 0$. This can be seen from evaluating the curl in cylindrical coordinates, or more easily by noting that

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} = 0,$$

since the current density $\mathbf{j}$ vanishes outside the wire.

2.4. Conservation laws for multiparticle systems.

For a system of N particles i, $1 \le i \le N$, the total momentum is

$$\mathbf{P} = \sum_i m_i \mathbf{v}_i,$$

where sums are understood to run from 1 to N. The total force on the system is the time rate of change of $\mathbf{P}$:

$$\mathbf{F}_{\text{tot}} = \frac{d\mathbf{P}}{dt} = \sum_i \mathbf{F}_i^{\text{ext}} + \sum_{i,j} \mathbf{F}^j_{\text{on } i}$$

where on the right-hand side we sum firstly over external forces on the particles, and secondly over forces on one particle from another.

However, by Newton's third law (action = reaction),

$$\mathbf{F}^j_{\text{on } i} = -\mathbf{F}^i_{\text{on } j}.$$

The last term is a sum over cancelling pairs and therefore vanishes. Then

$$\mathbf{F}_{\text{ext}} \equiv \sum_i \mathbf{F}_i^{\text{ext}} = \frac{d}{dt}\sum_i m_i \mathbf{v}_i = \frac{d^2}{dt^2}\sum_i m_i \mathbf{r}_i.$$

Defining

$$M \equiv \sum_i m_i$$
$$\mathbf{R} \equiv \frac{1}{M}\sum_i m_i \mathbf{r}_i,$$

we obtain

$$\mathbf{F}_{\text{ext}} = M\frac{d^2\mathbf{R}}{dt^2}. \tag{2.3}$$

That is, the motion of the center of mass coordinate $\mathbf{R}$ of a system of particles depends only on the total external force $\mathbf{F}_{\text{ext}}$, as if the system were merely a single particle located at that coordinate. If the total external force is zero, the velocity of the center of mass is constant.

Using the center of mass coordinate makes further simplifications possible, in the form of *decomposition theorems*. Consider the total angular momentum

$$\mathbf{L} = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i.$$

Substitute

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}_i^*$$
$$\mathbf{v}_i = \frac{d\mathbf{r}_i}{dt} = \frac{d\mathbf{R}}{dt} + \frac{d\mathbf{r}_i^*}{dt},$$

where $\mathbf{r}_i^*$ is the coordinate of particle i with respect to the center of mass coordinate $\mathbf{R}$:

$$\mathbf{L} = \sum_i (\mathbf{R} + \mathbf{r}_i^*) \times m_i(\frac{d\mathbf{R}}{dt} + \frac{d\mathbf{r}_i^*}{dt}).$$

Since

$$\sum m_i \mathbf{r}_i^* \equiv 0,$$

the two cross terms vanish, and

$$\mathbf{L} = \mathbf{R} \times \mathbf{P} + \sum_i \mathbf{r}_i^* \times \mathbf{p}_i^*. \tag{2.4}$$

That is, the total angular momentum is equal to the angular momentum *of* the center of mass plus the angular momentum *with respect to* the center of mass.

A similar proof yields the decomposition theorem for the kinetic energy:

$$T = \tfrac{1}{2}M(\frac{d\mathbf{R}}{dt})^2 + \sum_i \tfrac{1}{2}m_i(\frac{d\mathbf{r}_i^*}{dt})^2. \tag{2.5}$$

The total kinetic energy is equal to the kinetic energy of the center of mass plus the kinetic energy with respect to the center of mass.

2.5. Gravitational potential due to a spherically symmetric mass distribution.

The gravitational force on a point test particle of mass m_t due to another point mass m is, as usual,

$$\mathbf{F} = -\frac{Gmm_t}{r^2}\hat{e}_r,$$

where $\mathbf{r}$ is a vector from m (put at the origin) to m_t. To understand the force on m_t due to a mass *distribution*, it is useful to consider Gauss's theorem for the surface integral of $\mathbf{F}$:

$$\oiint \mathbf{F}\cdot d\mathbf{A} = \iiint (\nabla\cdot\mathbf{F})dv \qquad (2.6)$$

The radial part of the divergence in spherical coordinates is

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \ldots$$

so that $\nabla\cdot\mathbf{F}$ vanishes except at the origin, where it is singular. The spherical symmetry of the gravitational force makes the left-hand side of (2.6) easy to evaluate over a sphere of radius r:

$$\begin{aligned}\oiint \mathbf{F}\cdot d\mathbf{A} &= 4\pi r^2 \frac{-Gmm_t}{r^2}\\ &= -4\pi Gmm_t = \iiint (\nabla\cdot\mathbf{F})dv\end{aligned}$$

The last equality states that $\nabla\cdot\mathbf{F}$, already shown to be infinite at the origin, has a finite volume integral. This means that it is proportional to a 3-dimensional δ function $\delta^3(\mathbf{r})$, whose volume integral is defined to be unity:

$$\nabla\cdot\mathbf{F} = -4\pi Gmm_t\delta^3(\mathbf{r}).$$

In the case of a mass *distribution* instead of a mass *point*, the quantity $m\delta^3(\mathbf{r})$, which has dimensions mass per unit volume, is replaced by $\rho(\mathbf{r})$, the *mass density*. Equation (2.6) is replaced by

$$\begin{aligned}\oiint \mathbf{F}\cdot d\mathbf{A} &= \iiint (-4\pi Gm_t\rho(\mathbf{r}))dv.\\ &= -4\pi Gm_t M,\end{aligned} \qquad (2.7)$$

using the fact that the integral of ρ over the volume is just the mass M inside.

If the mass distribution $\rho(\mathbf{r})$ is spherically symmetric, we can use a spherical surface of radius r again to evaluate the left-hand side of Eq. (2.7). It becomes:

$$\begin{aligned}4\pi r^2 F_r(r) &= -4\pi Gm_t M\\ \mathbf{F}_{\rm grav} &= -\hat{e}_r\frac{GMm_t}{r^2}\end{aligned} \qquad (2.8)$$

This proves (without any messy integrations) that the gravitational force due to a spherically symmetric mass distribution is the same as that from a point at the origin with the same mass, provided that the force is observed outside the mass distribution.

The *gravitational field* $\mathbf{g}$ is defined as the gravitational force $\mathbf{F}_{\rm grav}$ divided by the test mass m_t. Thus it is analagous to the electrostatic field in that its value is independent of the constant (test mass or test charge) with which the test particle couples to the field. Again, if the source of the gravitational field is a spherically symmetric distribution of total mass M,

$$\mathbf{g} = -\hat{e}_r\frac{GM}{r^2}.$$

The potential $U_{\rm grav}$ from which this gravitational field is derived, defined by $\mathbf{g}\equiv -\nabla U_{\rm grav}$, is

$$U_{\rm grav} = -\frac{GM}{r},$$

adopting the convention that $U_{\rm grav} = 0$ at $r=\infty$.

3. Oscillations.

3.1. Differential equation for linear oscillations.

Any potential energy $U(x')$ can be expanded about a minimum, for example at $x'=a$:

$$\begin{aligned}U(x') = U(a) + (x'-a)\frac{\partial U}{\partial x'}\Big|_a +&\\ + \tfrac{1}{2}(x'-a)^2\frac{\partial^2 U}{\partial x'^2}\Big|_a + \ldots&\end{aligned}$$

Redefining $U(a)\equiv 0$, introducing $x\equiv x'-a$, and expressing the (single-particle) kinetic energy as $T=\frac{1}{2}m\dot{x}^2$, energy conservation demands

$$\begin{aligned}E = \text{constant} = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2,\ k\equiv \frac{\partial^2 U}{\partial x^2}\Big|_0\\ \frac{dE}{dt} = 0 = \tfrac{1}{2}m(2\dot{x}\ddot{x}) + \tfrac{1}{2}k(2x\dot{x}),\end{aligned}$$

with the trivial solution $\dot{x} = 0$ and the nontrivial solution

$$m\ddot{x} + kx = 0.$$

Therefore the familiar "mass-spring" force equation is obtained for any potential that has a minimum, provided that the excursions are kept small.

Adding a viscous damping force and a driving force,

$$F_x^{\text{damp}} = -b\dot{x}$$
$$F_x^{\text{drive}} = F_0 \cos\omega t,$$

and defining $\gamma \equiv b/m$, $\omega_0^2 \equiv k/m$, one obtains the simple form

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = (F_0/m)\cos\omega t. \tag{3.1}$$

This is the "full" differential equation, in the sense that it has a driving term on the right-hand side (in the absence of a driving term, it would be called "homogeneous"). We seek a *particular* solution x_p of the full equation which is nontrivial. In order to match boundary conditions, later we will add a *general* solution x_h of the homogeneous equation. It will turn out that x_h vanishes at sufficiently large times; therefore x_p is the asymptotic solution as $t \to \infty$.

3.2. Particular solution.

To avoid unnecessary algebra in solving this equation, we will employ the *complex exponential method*. The first step is to substitute

$$x_p = \Re(\tilde{A}e^{i\omega' t})$$
$$F_0\cos\omega t = \Re(F_0 e^{i\omega t}),$$

where ω' is some as-yet-undetermined trial frequency, and $\tilde{A}$ is complex to allow for differences in phase between the driving force and the response. We *hope* that it is possible to find a solution for which $\tilde{A}$ is time-independent, a hope that will be fulfilled in this case. With these substitutions, (3.1) is

$$\begin{aligned} &-\omega'^2\Re(\tilde{A}e^{i\omega' t}) + \gamma\omega'\Re(i\tilde{A}e^{i\omega' t}) + \\ &+ \omega_0^2\Re(\tilde{A}e^{i\omega' t}) = \Re((F_0/m)e^{i\omega t}). \end{aligned} \tag{3.2}$$

As the second step, to avoid unnecessary algebra, we *choose* to solve the complex equation of which (3.2) is the real part, rather than (3.2) itself:

$$\begin{aligned} &-\omega'^2\tilde{A}e^{i\omega' t} + \gamma\omega' i\tilde{A}e^{i\omega' t} + \\ &+ \omega_0^2\tilde{A}e^{i\omega' t} = (F_0/m)e^{i\omega t}. \end{aligned} \tag{3.3}$$

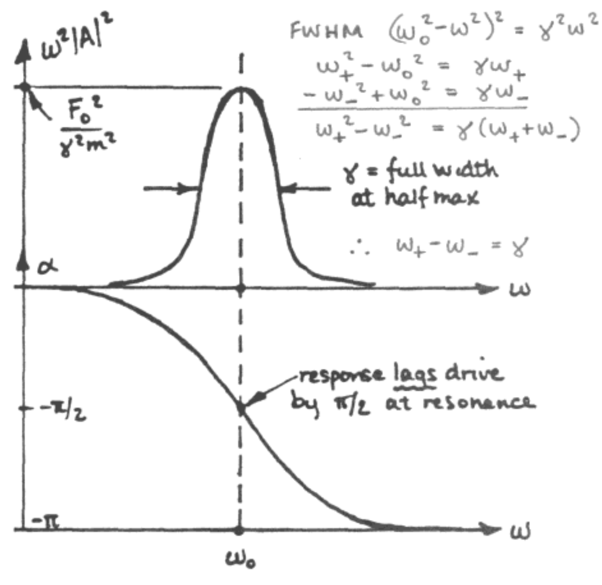
A nontrivial solution is impossible unless $\omega' = \omega$. Then the $e^{i\omega t}$ factors cancel, and

$$\tilde{A} = \frac{F_0/m}{(\omega_0^2 - \omega^2 + i\gamma\omega)}.$$

This is written in terms of a magnitude and a phase as

$$\begin{aligned} \tilde{A} &= \frac{F_0/m}{[(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2]^{1/2}}e^{i\alpha} \\ &\equiv |\tilde{A}|e^{i\alpha} \\ \alpha &= -\arctan\frac{\gamma\omega}{\omega_0^2-\omega^2}. \end{aligned} \tag{3.4}$$

This result has the following properties: $\omega^2|\tilde{A}|^2$ (which is proportional to the oscillator's average rate of energy dissipation) reaches a peak of value $F_0^2/\gamma^2 m^2$ at the resonant frequency $\omega = \omega_0$. The half maximum of $\omega^2|\tilde{A}|^2$ occurs when $(\omega_2^2 - \omega_0^2) = +\gamma\omega_2$ or $(\omega_1^2 - \omega_0^2) = -\gamma\omega_1$. It follows easily that the full width at half maximum ("FWHM") of the resonant peak is $\omega_2 - \omega_1 = \gamma$.



As for the phase, assuming

$$Q \equiv \frac{\omega_0}{\gamma} \gg 1,$$

α starts just below zero at low frequency, falls through $-\pi/2$ at resonance, and approaches $-\pi$ at high frequency. The abruptness with which α crosses $-\pi/2$ increases as the resonance gets sharper (i.e. as the "quality factor" Q gets larger; note that Q increases when the damping decreases). In other words, the response $\tilde{A}$ *lags* the driving force by very little at low frequencies, by 90° at resonance, and by nearly 180° at high frequencies. The final step is to substitute back for x_p:

$$\begin{aligned} x_p &= \Re(\tilde{A}e^{i\omega t}) \\ &= \Re(|\tilde{A}|e^{i\alpha}e^{i\omega t}) \\ &= |\tilde{A}|\cos{(\omega t + \alpha)} \end{aligned} \tag{3.5}$$

with $|\tilde{A}|$ and α as in (3.4).

In some texts, confusion is spread by analyzing the peak in $|\tilde{A}|$ rather than in $\omega^2|\tilde{A}|^2$. (The latter is physically the more meaningful quantity, as it is proportional to the power dissipated in the oscillator.) This confusion leads to messy nonstandard definitions for the resonant frequency and for the quality factor, which are best ignored.

3.3. Homogeneous solutions.

Having found a nontrivial particular solution x_p to the full equation, we turn to the general solution x_h to the homogeneous equation. Since the right-hand side of the homogeneous equation is zero, it is clear that the sum of x_p and x_h will still satisfy the full equation. Taking this sum is the easiest way to obtain a general solution to the full equation.

The homogeneous equation is (3.1) without the driving term:

$$\ddot{x}_h + \gamma\dot{x}_h + \omega_0^2 x_h = 0. \tag{3.6}$$

Using the same complex exponential method, and ignoring the trivial solution $\tilde{A} = 0$, we obtain

$$-\omega^2 + i\gamma\omega + \omega_0^2 = 0.$$

The solutions are given by the quadratic formula:

$$\omega_\pm = \frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \equiv \frac{i\gamma}{2} \pm \omega_\gamma.$$

The discriminant, which can be written as $\gamma^2(Q^2 - \frac{1}{4})$, distinguishes three cases: (*i*) $Q > \frac{1}{2}$ ("underdamped"); (*ii*) $Q < \frac{1}{2}$ ("overdamped"); (*iii*) $Q = \frac{1}{2}$ ("critically damped").

In the *underdamped* case, the discriminant is positive and ω_γ is real. The general solution is an arbitrarily weighted sum of the two particular solutions involving ω_+ and ω_-:

$$x_h = e^{-\gamma t/2}\Re(\tilde{A}_+ e^{i\omega_\gamma t} + \tilde{A}_- e^{-i\omega_\gamma t}).$$

The last factor can be written as a cosinusoid of $\omega_\gamma t$ within a phase:

$$x_h = Be^{-\gamma t/2}\cos{(\omega_\gamma t + \beta)}, \tag{3.7}$$

where the amplitude B and phase β are adjusted to fit the boundary conditions.

In the *overdamped* case, the discriminant is negative and ω_γ is imaginary. Defining

$$\gamma_\pm \equiv \frac{\omega_\pm}{i} = \frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2},$$

x_h again is an arbitrarily weighted sum of the two particular solutions:

$$x_h = C_+ e^{-\gamma_+ t} + C_- e^{-\gamma_- t}. \tag{3.8}$$

Note that if $\gamma/2 \gg \omega_0$, or $Q \ll 1$ (ultradamping), γ_+ is close to γ while γ_- is much smaller.

Finally, in the *critically damped* case, never exactly achieved in practice, $Q \equiv \frac{1}{2}$, the discriminant vanishes, and

$$\begin{aligned} \omega_+ = \omega_- &= i\gamma/2, \\ x_h &= \Re(\tilde{D}e^{-\gamma t/2}). \end{aligned} \tag{3.9}$$

This is only one solution; to get two adjustable constants, necessary for a second-order differential equation, we must find another. This is accomplished by writing (3.6), with $\gamma = 2\omega_0$, as

$$\Big(\frac{d}{dt} + \omega_0\Big)\Big(\frac{d}{dt} + \omega_0\Big)x_h = 0. \tag{3.10}$$

The solution (3.9) is a solution to

$$(\frac{d}{dt}+\omega_0)x_h = (\frac{d}{dt}+\omega_0)e^{-\omega_0 t} = 0.$$

If we had another solution $s(t)$ such that

$$(\frac{d}{dt}+\omega_0)s(t) = e^{-\omega_0 t},$$

then $s(t)$ would also solve (3.10). Substituting the trial solution $s(t) = f(t)\exp{(-\omega_0 t)}$, we easily find $f(t) = t$. Then the general solution to the critically damped case is

$$x_h = D_1 e^{-\gamma t/2} + D_2 t e^{-\gamma t/2}. \tag{3.11}$$

This is the transient response that designers of mechanical ammeters, shock absorbers, etc., attempt to achieve: it approaches the asymptotic state as quickly as possible without overshoot. Despite the brisk market in heavy-duty shocks, putting them on your car merely moves the suspension from critically damped to overdamped; the ride becomes harsher but not better controlled.

Once the solutions for these three cases are developed, the applications tend to be routine. Most fall in two classes: (i) steady-state response to a periodic driving waveform, and (ii) transient response to a non-periodic driving waveform, i.e. a switch closing. Only occasionally is one asked to combine (i) and (ii). If this happens, remember to match boundary conditions using the *sum* of the particular and homogeneous solutions $x_p(t) + x_h(t)$, rather than using only the latter.

3.4. Fourier expansion of the driving term.

In problems of class (i), if the driving waveform is (co)sinusoidal the problem is already solved, by (3.5). As an example of the solution of problems of this class, we consider a non-sinusoidal driving force $f(t)$ with period T and zero average value. This problem is solved by expanding $f(t)$ in a *Fourier series* of sines and/or cosines:

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty} f_n \cos\omega_n t + g_n \sin\omega_n t; \\ \omega_n &\equiv \frac{2\pi n}{T}. \end{aligned} \tag{3.12}$$

The constants f_n and g_n can be found using *Fourier's trick*. For example, to find f_m, where $1 \le m \le \infty$, multiply (3.12) by $(2/T)\cos 2\pi mt/T$, and integrate over one period:

$$\begin{aligned} &\frac{2}{T}\int_0^T dt\,\cos\frac{2\pi mt}{T} f(t) \;= \\ &= \frac{2}{T}\sum_{n=1}^{\infty}\Bigl(f_n\int_0^T dt\,\cos\frac{2\pi mt}{T}\cos\frac{2\pi nt}{T} + \\ &\qquad + g_n\int_0^T dt\,\cos\frac{2\pi mt}{T}\sin\frac{2\pi nt}{T}\Bigr). \end{aligned}$$

Using the orthonormality of the cosines and sines,

$$\begin{aligned} \frac{2}{T}\int_0^T dt\,\cos\frac{2\pi mt}{T}\cos\frac{2\pi nt}{T} &= \delta_{mn} \\ \frac{2}{T}\int_0^T dt\,\sin\frac{2\pi mt}{T}\sin\frac{2\pi nt}{T} &= \delta_{mn} \\ \frac{2}{T}\int_0^T dt\,\cos\frac{2\pi mt}{T}\sin\frac{2\pi nt}{T} &= 0, \end{aligned}$$

all terms on the right-hand side vanish except the $\cos^2$ term with $n = m$. Then (and similarly for g_m),

$$\begin{aligned} f_m &= \frac{2}{T}\int_0^T dt\,\cos\frac{2\pi mt}{T} f(t) \\ g_m &= \frac{2}{T}\int_0^T dt\,\sin\frac{2\pi mt}{T} f(t). \end{aligned} \tag{3.13}$$

Since the oscillator is linear, the solution to a sum of driving terms is the sum of the individual solutions:

$$\begin{aligned} x_p = \sum_{n=1}^{\infty} |\tilde{a}_n|[&f_n\cos{(\omega_n t + \alpha_n)} + \\ &+ g_n\sin{(\omega_n t + \alpha_n)}], \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} |\tilde{a}_n| &= \frac{1/m}{[(\omega_0^2 - \omega_n^2)^2 + \gamma^2\omega_n^2]^{1/2}} \\ \alpha_n &= -\arctan\frac{\gamma\omega_n}{\omega_0^2 - \omega_n^2}. \end{aligned}$$

3.5. Response of the underdamped oscillator to a δ-function drive.

As an example of a problem of type (*ii*), we consider the response of an oscillator with $Q > \frac{1}{2}$ to a δ-function driving term. An infinite force F_δ is applied for an infinitesimal time such that its time integral is a constant mv_0:

$$\begin{aligned} F_\delta(t) &= 0 \ (t \neq 0); \\ F_\delta(t) &= \infty \ (t = 0); \\ \int_{-\infty}^{\infty} \frac{F_\delta(t)}{m}\, dt &= v_0 \end{aligned} \tag{3.15}$$

This is equivalent to requiring

$$F_\delta(t) = m v_0 \delta(t),$$

where $\delta(t)$ is a one-dimensional Dirac delta function like the three-dimensional type discussed in section 2.5. The action of this force may be simulated by striking a resting mass at $t = 0$ with a hard object so that the mass obtains a velocity v_0. What are the boundary conditions?

Within the infinitesimal time interval that $F_0(t)$ is nonzero, it is so large that all the other forces are negligible in comparison to it. Therefore, after that interval, the mass will acquire the velocity v_0. However, during the same interval, the displacement $\int v\, dt$ of the mass still is infinitesimal, because v is finite. Taking the origin of coordinates at the resting position, the boundary conditions become

$$0 = x(0^+); \;\; v_0 = \dot{x}(0^+).$$

Using Eq. (3.7),

$$\begin{aligned} 0 &= B \cos\beta \\ v_0 &= -\frac{\gamma}{2} B \cos\beta - \omega_\gamma B \sin\beta. \end{aligned}$$

Solving for the constants,

$$\begin{aligned} \beta &= \pi/2 \\ B &= \frac{-v_0}{\omega_\gamma}. \end{aligned}$$

Plugging these constants into (3.7) for the solution,

$$\begin{aligned} x(t) &= 0 \;\; (t < 0); \\ &= v_0 \frac{e^{-\gamma t/2} \sin \omega_\gamma t}{\omega_\gamma} \;\; (t > 0); \\ (\omega_\gamma^2 &\equiv \omega_0^2 - \gamma^2/4). \end{aligned} \tag{3.16}$$

3.6. Green function for the underdamped oscillator.

For an underdamped oscillator initially at rest, the solution (3.16) is just the *Green function* $G(t)$ multiplied by v_0. Associated with many homogeneous differential equations and boundary conditions are unique Green functions. If the Green function is known, the solution to the differential equation, in the presence of *any* driving term, may be found by performing a single integration.

Consider a linear differential operator D_t (in the case just considered, $D_t = d^2/dt^2 + \gamma\, d/dt + \omega_0^2$). The Green function $G(t)$ is defined to be the solution to the equation

$$D_t G(t) \equiv \delta(t).$$

Generalizing to a delta-function that peaks at $t = t'$ rather than $t = 0$,

$$D_t G(t, t') \equiv \delta(t - t').$$

For an arbitrary driving term $a(t)$, the solution to the differential equation $D_t x = a(t)$ is the integral

$$x(t) = \int_{-\infty}^{\infty} G(t, t') a(t')\, dt', \tag{3.17}$$

as is easily verified:

$$\begin{aligned} D_t x &= \int_{-\infty}^{\infty} D_t G(t, t') a(t')\, dt' \\ &= \int_{-\infty}^{\infty} \delta(t - t') a(t')\, dt' \\ &= a(t). \end{aligned}$$

In the last step we used the fundamental property of the Dirac δ function

$$\int_{-\infty}^{\infty} \delta(t - t') f(t')\, dt' = f(t)$$

for any function f. That is, the (rest of the) integrand is evaluated where the δ function becomes infinite.

Returning to the underdamped oscillator, for any driving term $a(t)$ the solution obtained with the help of the Green function is

$$x(t) = \int_{-\infty}^{t} \frac{e^{-\gamma(t-t')/2} \sin \omega_\gamma (t-t')}{\omega_\gamma} a(t')\, dt',$$

provided that the mass is at rest at the origin before the driving force starts. We wrote the upper limit of the integral as t rather than ∞ because G vanishes for $t < t'$.

For a practical application in which determining the answer with adequate numerical precision is the main objective, a solution in the form of an integral is wholly acceptable. The integral may be evaluated to arbitrarily high precision using a digital computer.

You already know at least one other Green function. In section 2.5 we found that the gravitational potential U from a point mass m is $U = -Gm/r$. The gravitational force $\mathbf{F} = -m_t \nabla U$, where m_t is a test mass, satisfies

$$\nabla \cdot \mathbf{F} = -4\pi G m m_t \delta^3(\mathbf{r}).$$

Then

$$\begin{aligned} -\nabla^2 \big(-m_t \frac{Gm}{r}\big) &= -4\pi G m m_t \delta^3(\mathbf{r}) \\ \nabla^2 \frac{1}{r} &= -4\pi \delta^3(\mathbf{r}) \\ \nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} &= -4\pi \delta^3(\mathbf{r}-\mathbf{r}'), \end{aligned}$$

where we have generalized the last equation to allow the mass point to be located at any coordinate $\mathbf{r}'$. This demonstrates that the Green function $G(\mathbf{r}, \mathbf{r}')$ for the differential operator ∇^2 is

$$G(\mathbf{r}, \mathbf{r}') = \frac{-1}{4\pi |\mathbf{r}-\mathbf{r}'|},$$

subject to the boundary condition $G(\infty, \mathbf{r}') = 0$. Correspondingly, a differential equation of the general form

$$\nabla^2 f(\mathbf{r}) = a(\mathbf{r}),$$

where a is any driving term, has the Green function solution

$$f(\mathbf{r}) = \iiint \frac{-a(\mathbf{r}')}{4\pi |\mathbf{r}-\mathbf{r}'|} d^3 r',$$

where the integral is taken over all space.

3.7. Nonlinear oscillations.

In general, oscillatory motion occurs (for example, in one dimension x) when the total energy E in a conservative system exceeds $U(x)$ only within a finite region. Within this region, say $x_A < x < x_B$, the kinetic energy T is positive. At $x = x_A$ and $x = x_B$, T vanishes; x_A and x_B are the *classical turning points*.

The period of the motion may be determined by a simple integral. Starting from the equation of energy conservation,

$$\begin{aligned} T &= \tfrac{1}{2} m \dot{x}^2 = E - U(x) \\ \frac{dx}{dt} &= \big\{ \frac{2}{m} [E - U(x)] \big\}^{1/2} \\ \frac{T_0}{2} &= \int_{t_A}^{t_B} dt \\ &= \int_{x_A}^{x_B} \frac{dx}{\big\{ \frac{2}{m} [E - U(x)] \big\}^{1/2}}. \end{aligned} \tag{3.18}$$

For simple harmonic motion, with $U(x) = \frac{1}{2} k x^2$ and $A = -B$, the usual period $T = 2\pi \sqrt{m/k}$ is easily obtained by doing the integral. On the other hand, for a pendulum oscillating through angles that are not negligibly small, the integral for the period is not elementary. With θ substituted for x, I for m, and $U(\theta) = mgl(1 - \cos \theta)$, it is an *elliptic integral of the first kind*. This integral may be evaluated by using tables, or by Taylor-series expanding $(1 - \cos \theta)$. As expected, to lowest order the Taylor expansion merely recovers the simple-harmonic-motion period. As a practical matter, fast digital computation usually obviates the need for tables or series expansions in evaluating any of these integrals.

In addition to obtaining an integral solution for the period, one may solve the differential equation for nonlinear oscillation if it is *only slightly* nonlinear. The procedure used, the *method of perturbations*, is more interesting than this problem alone, because it is a fundamental method of classical and especially of modern theoretical physics.

Consider an undamped linear oscillator under the influence of a small additional nonlinear force $m\lambda x^2$:

$$\ddot{x} + \omega_0^2 x - \lambda x^2 = 0.$$

Here "small" means that $|\lambda x| \ll \omega_0^2$. Since λ is small, we attempt to find a solution of the form

$$\begin{aligned} x(t) &= x_0(t) + \lambda\eta(t) \\ 0 &= \ddot{x}_0 + \omega_0^2 x_0 - \lambda x_0^2 + \lambda\ddot{\eta} + \omega_0^2\lambda\eta + \\ &\quad + \text{ (terms of higher order in } \lambda). \end{aligned} \tag{3.19}$$

Since λ, though small, is a constant of otherwise arbitrary size, x_0 must solve the simple harmonic equation $\ddot{x}_0 + \omega_0^2 x_0 = 0$. Assuming the boundary condition $\dot{x}_0(0) = 0$, the solution for x_0 is

$$x_0(t) = A\cos\omega_0 t.$$

Plugging x_0 back into (3.19), the first two terms vanish. Neglecting the higher-order terms,

$$\begin{aligned} \lambda(-A^2\cos^2\omega_0 t + \ddot{\eta} + \omega_0^2\eta) &= 0 \\ \ddot{\eta} + \omega_0^2\eta &= \frac{A^2}{2}(\cos 2\omega_0 t + 1), \end{aligned}$$

using the relation $2\cos^2 y = \cos 2y + 1$.

Ignoring the constant driving term on the right-hand side, the last equation is the same as (3.1) with $\gamma = 0$. The particular solution is (3.5) with $\gamma = 0$:

$$\eta = \frac{\frac{A^2}{2}\cos 2\omega_0 t}{\omega_0^2 - (2\omega_0)^2} + \frac{A^2}{2\omega_0^2}.$$

The last constant is added to satisfy the constant driving term. Simplifying,

$$\eta = \frac{A^2}{2\omega_0^2}\left(1 - \tfrac{1}{3}\cos 2\omega_0 t\right).$$

It is characteristic of the solution that η is proportional to a *power* of the unperturbed amplitude A (in this case the square). It is also characteristic that the presence of the nonlinear term causes a perturbation to the response which occurs at a *harmonic* of the fundamental frequency ω_0, in this case the second harmonic. In general, departures from linearity cause an oscillator to exhibit *harmonic distortion*, as is all too obvious in a loudspeaker that is driven too hard.

The perturbation solutions to slightly nonlinear oscillators are rarely as straightforward as in this example. Often η is found to contain terms that increase indefinitely with t, so that $\lambda\eta$ eventually cannot remain small. Nevertheless, the main point of this example is to introduce the method of perturbations. Recapitulating, to solve a standard problem when a small additional force or potential is introduced, add a small new term to the standard solution, plug into the differential equation, retain terms to first order in smallness, take advantage of the cancellations, and solve for the new term.

4. Calculus of variations.

4.1. Euler equation.

Initially we focus on the purely mathematical problem of finding the *shape* of a curve $y(t)$, where t is an independent variable not necessarily equal to the time, such that the quantity

$$J \equiv \int_{t_1}^{t_2} \mathcal{L}(y, \dot{y}, t)\,dt$$

is *stationary*. Here $\mathcal{L}$ is an arbitrary, continuously differentiable function of the indicated variables. In other words, we seek a path $y(t)$ such that the integral of $J = \mathcal{L}(y, \dot{y}, t)$ doesn't vary as the path is varied infinitesimally. Usually this means that J is minimized or maximized.

Consider all possible paths between (t_1, y_1) and (t_2, y_2). To make the problem more specific, consider only the subset of paths which begin at *fixed* $y_1(t_1)$ and end at *fixed* $y_2(t_2)$. We parameterize these paths by a single variable α such that, by convention, $\alpha = 0$ for the "best" path, i.e. the path which produces a stationary value for the above integral. We label the various paths by α:

$$J_\alpha = \int_{t_1}^{t_2} \mathcal{L}\big(y_\alpha(t), \dot{y}_\alpha(t), t\big)\, dt,$$
$$J(\alpha) = \int_{t_1}^{t_2} \mathcal{L}\big(y(\alpha,t), \dot{y}(\alpha,t), t\big)\, dt.$$

In the latter expression we have chosen to consider J, y, and $\dot{y}$ to be functions of the label α. Both notations have the same meaning; in the following we shall use the latter.

The requirement that α vanish when J is stationary means that

$$\begin{aligned} \frac{\partial J}{\partial \alpha}\Big|_{\alpha=0} &= 0, \text{ where} \\ \frac{\partial J}{\partial \alpha} &= \int_{t_1}^{t_2} \Big[\frac{\partial \mathcal{L}}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\Big]\, dt. \end{aligned} \tag{4.1}$$

Performing a parts integration on the second term,

$$\int u dv = uv - \int v du$$
$$u = \frac{\partial \mathcal{L}}{\partial \dot{y}} \quad du = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} dt$$
$$dv = \frac{\partial^2 y}{\partial t \partial \alpha} dt \quad v = \frac{\partial y}{\partial \alpha}$$
$$\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\, dt = \frac{\partial \mathcal{L}}{\partial \dot{y}}\frac{\partial y}{\partial \alpha}\Big|_{t_1}^{t_2} - $$
$$- \int_{t_1}^{t_2} \frac{\partial y}{\partial \alpha}\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}\, dt.$$

The first term on the right-hand side vanishes because, by assumption, $y(t_1)$ and $y(t_2)$ are the same for every α. Finally,

$$\frac{\partial J}{\partial \alpha} = \int_{t_1}^{t_2} \frac{\partial y}{\partial \alpha}\Big[\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}\Big]\, dt.$$

The next step is to multiply through by an arbitrary small displacement $\delta\alpha$ and evaluate the derivatives with respect to α at $\alpha = 0$:

$$\frac{\partial J}{\partial \alpha}\Big|_{\alpha=0}\delta\alpha = \int_{t_1}^{t_2} \frac{\partial y}{\partial \alpha}\Big|_{\alpha=0}\delta\alpha\Big[\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}\Big]\, dt.$$

Defining

$$\frac{\partial J}{\partial \alpha}\Big|_{\alpha=0}\delta\alpha \equiv \delta J$$
$$\frac{\partial y}{\partial \alpha}\Big|_{\alpha=0}\delta\alpha \equiv \delta y,$$

we have

$$\delta J = \int_{t_1}^{t_2} \delta y\Big[\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}\Big]\, dt. \tag{4.2}$$

We are now in a position to make the final argument. Because we are considering all possible paths between (y_1, t_1) and (y_2, t_2), δy is a *completely arbitrary* displacement at each point on the path. δJ can vanish *only if* the part of the integrand that multiplies δy also vanishes, i.e.

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} \\ \mathcal{L} &= \mathcal{L}(y, \dot{y}, t). \end{aligned} \tag{4.3}$$

This is the celebrated *Euler equation.* When $\mathcal{L}$ is equal to the *Lagrangian* $T - U$, J is called the *action* and the Euler equation becomes the *Euler-Lagrange equation.*

4.2. Example using Euler equation.

As an example of the use of the Euler equation, again considering a purely mathematical problem, we *minimize the surface of revolution.* The problem is easier to visualize if we temporarily change the notation. Denote the independent variable, usually t, as the (cylindrical) radius r; and denote the generalized coordinate, usually y, as the distance z along the (cylindrical) axis. In this notation, the Euler equation is

$$0 = \frac{\partial \mathcal{L}}{\partial z} - \frac{d}{dr}\frac{\partial \mathcal{L}}{\partial \dot{z}}$$
$$\mathcal{L} = \mathcal{L}(z, \dot{z}, r).$$

In this notation, the problem asks us to find the curve $z(r)$ with fixed endpoints $z(r_1) = z_1$, $z(r_2) = z_2$, such that the surface of revolution about the z axis has minimum area.

Along the curve $z(r)$, the path length is

$$\begin{aligned} ds &= [(dr)^2 + (dz)^2]^{1/2} \\ &= dr[1 + \dot{z}^2]^{1/2}, \end{aligned}$$

where $\dot{z}$ means differentiation of z with respect to the independent variable r. When rotated about the z axis, this element of path length produces an element of surface area

$$dA = 2\pi r[1 + \dot{z}^2]^{1/2} dr.$$

So the problem reduces to minimizing the integral

$$\int_{r_1}^{r_2} r[1 + \dot{z}^2]^{1/2} dr,$$

subject to the condition that the endpoints z_1 and z_2 are fixed. Then the integrand is

$$\mathcal{L}(z, \dot{z}, r) = r[1 + \dot{z}^2]^{1/2}.$$

Applying the Euler equation,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial z} = 0 &= \frac{d}{dr}\frac{\partial \mathcal{L}}{\partial \dot{z}} \\ &= \frac{d}{dr}\frac{\dot{z}r}{[1 + \dot{z}^2]^{1/2}} \\ r_0 = \text{const} &= \frac{\dot{z}r}{[1 + \dot{z}^2]^{1/2}} \\ r_0^2(1 + \dot{z}^2) &= \dot{z}^2 r^2 \\ r_0^2 &= \dot{z}^2(r^2 - r_0^2) \\ \frac{dz}{dr} &= \frac{r_0}{(r^2 - r_0^2)^{1/2}} \\ z &= r_0 \int \frac{dr}{(r^2 - r_0^2)^{1/2}} \\ &= r_0 \cosh^{-1}\frac{r}{r_0} + z_0 \\ r &= r_0 \cosh\frac{z - z_0}{r_0}, \end{aligned}$$

where r_0 and z_0 are constants determined by z_1 and z_2. This is the equation of a *catenary* (the shape of the cables on a suspension bridge).

4.3. Equations of constraint.

An *equation of constraint* is an *additional* equation introduced to constrain the generalized coordinate $y(t)$. Suppose, as before, we consider the problem of making stationary the action J when the integrand $\mathcal{L}$ is a function only of a *single* coordinate y, its time derivative $\dot{y}$, and the independent variable t. In this simplest case it would be foolish to impose an equation of constraint, because the constraint would determine $y(t)$ by itself, and there would be nothing left to make stationary. Therefore, equations of constraint are relevant only when $\mathcal{L}$ is a function of two or more coordinates.

Suppose that $\mathcal{L} = \mathcal{L}(y, z, \dot{y}, \dot{z}, t)$. Temporarily, we will assume that no constraint equations apply. Under these circumstances, the derivation of the Euler equations proceeds much the same way as in section 4.1. The integrand in Eq. (4.1) acquires *four* terms, because $\mathcal{L}$ must be differentiated with respect to y, z, $\dot{y}$, and $\dot{z}$. Equation (4.2) becomes

$$\begin{aligned} \delta J = \int_{t_1}^{t_2} \Big\{ \delta y \Big[\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}\Big] + \\ + \delta z \Big[\frac{\partial \mathcal{L}}{\partial z} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}}\Big]\Big\} dt. \end{aligned} \tag{4.4}$$

Since each of the virtual displacements δy and δz are independent, two Euler equations must be satisfied for δJ to vanish:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} \\ 0 &= \frac{\partial \mathcal{L}}{\partial z} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}} \\ \mathcal{L} &= \mathcal{L}(y, z, \dot{y}, \dot{z}, t). \end{aligned} \tag{4.5}$$

For n generalized coordinates, one obtains n equations of the same form.

Now that we understand how to handle the case of two generalized coordinates when there is no equation of constraint, we return to the main problem. We are faced with finding a path $y(t), z(t)$ that makes stationary the path integral of $\mathcal{L}$ when the path's endpoints are fixed – while satisfying an additional equation that constrains y and z. Unfortunately, the constraint equation can take many forms. If it is a *differential* equation that is *not of first order*, the only hope for doing the problem is to solve the constraint equation, or at least reduce it to first order by changing variables. If the constraint equation is a first-order differential equation

$$g_y \frac{dy}{dt} + g_z \frac{dz}{dt} + g_t = 0, \tag{4.6}$$

where g_y, g_z, and g_t, like $\mathcal{L}$, are functions of y, z, $\dot{y}$, $\dot{z}$, and t, it can be solved by the method of *Lagrange undetermined multipliers*, which is the subject of the next section.

If the constraint equation is an *algebraic* rather than a differential equation, $G(y, z, t) = 0$, it is called *holonomic* and is merely a special case of Eq. (4.6), since

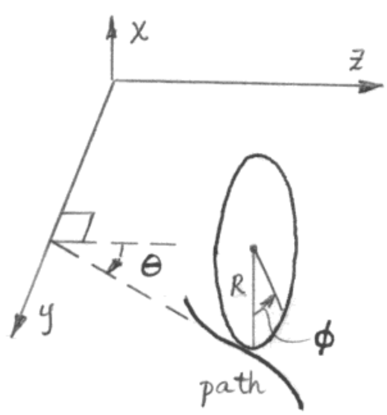
$$g_y \equiv \frac{\partial G}{\partial y} \quad g_z \equiv \frac{\partial G}{\partial z} \quad g_t \equiv \frac{\partial G}{\partial t}.$$

Finally, if the constraint equation takes the form

$$\int_{t_1}^{t_2} \mathcal{G}\, dt = \text{constant}, \tag{4.7}$$

where $\mathcal{G}$ is a function of the same variables as $\mathcal{L}$, the problem can also be solved by using an undetermined multiplier.

Before proceeding to discuss undetermined multipliers, it is instructive to identify the differential (nonholonomic) equation of constraint for a physically meaningful example. Consider a thin coin of radius R rolling upright on the plane $x = 0$, where $\hat{x}$ is up. The C.M. of the coin is at (y, z). Let ϕ be the azimuth of the coin, measured with respect to the point of contact, and let θ be the orientation of the face of the coin, measured with respect to the xz plane. As long as the coin remains upright, these four generalized coordinates define the position and orientation of the coin, and their four time derivatives completely describe its motion.



If the surface $x = 0$ is frictionless, there need be no equation of constraint relating the generalized coodinates. However, if there is friction, we might require that the coin be *rolling without slipping*. In that case, for a particular choice of sign for ϕ and θ, the constraint equations are:

$$-\dot{y} = R\dot{\phi}\sin\theta$$
$$-\dot{z} = R\dot{\phi}\cos\theta.$$

We cannot integrate and solve these equations without solving the whole problem. However, we can identify the coefficients g_i in the notation of Eq. (4.6). For these two equations of constraint, they are, respectively,

$$\begin{array}{lllll} g_y = 1 & g_z = 0 & g_\theta = 0 & g_\phi = R\sin\theta & g_t = 0 \\ g_y = 0 & g_z = 1 & g_\theta = 0 & g_\phi = R\cos\theta & g_t = 0. \end{array}$$

4.4. Method of Lagrange undetermined multipliers.

In Eq. (4.4), when y and z were two independent generalized coordinates, we argued that δy and δz were independently arbitrary. This yielded the two Euler equations in Eq. (4.5). However, the presence of an equation of constraint which links y and z destroys that independence. The constraint equation (4.6) links the two virtual displacements:

$$g_y \delta y + g_z \delta z = -g_t \delta t = 0.$$

The term on the right-hand side vanishes because the displacements occur at fixed time t for each point on the path $y(t), z(t)$.

Now we introduce the *Lagrange undetermined multiplier* $\lambda(t)$. For any λ it is obvious that

$$(g_y \delta y + g_z \delta_z)\lambda = 0.$$

Inserting this in Eq. (4.4),

$$\delta J = \int_{t_1}^{t_2} \Big\{ \delta y \Big[\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} + \lambda g_y\Big] + + \delta z \Big[\frac{\partial \mathcal{L}}{\partial z} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}} + \lambda g_z\Big] \Big\} dt. \tag{4.8}$$

Since δy and δz cannot be independent of each other, we *choose* δy to be the independent virtual displacement and δz to be the dependent one. This means that the first square bracket in Eq. (4.8) must vanish. Since λ is undetermined, we are free to *choose* λ so that the second square bracket in Eq. (4.8) vanishes too.

To summarize, the effect of introducing the equation of constraint and the Lagrange undetermined multiplier is to increase the number of unknown functions of t from two ($y(t)$ and $z(t)$) to three, with $\lambda(t)$ included; and, correspondingly, to increase the number of differential equations from two to three:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} + \lambda g_y \\ 0 &= \frac{\partial \mathcal{L}}{\partial z} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}} + \lambda g_z \\ 0 &= g_y \dot{y} + g_z \dot{z} + g_t, \end{aligned} \tag{4.9}$$

where the last equation is just a repetition of Eq. (4.6).

4.5. Lagrange multiplier applied to integral constraint.

Suppose the constraint equation is of the integral form (4.7). Since the integral in that equation is constant, as α is varied the virtual displacement of the integral is zero. This means that we can multiply the integrand $\mathcal{G}$ in (4.7) by a *constant* undetermined multiplier Λ and add it to the integrand $\mathcal{L}$ in the virtual displacement of the action:

$$0 = \delta J = \delta \int_{t_1}^{t_2} (\mathcal{L} + \Lambda \mathcal{G})\, dt.$$

The resulting set of Euler equations is the same as Eq. (4.5) with $\mathcal{L}$ replaced by $\mathcal{L} + \Lambda\mathcal{G}$. In their derivation, which we do not elaborate here, the addition of $\Lambda\mathcal{G}$ to $\mathcal{L}$ makes it possible to continue to regard the virtual displacements δy and δz as independent, even in the presence of the integral equation of constraint.

4.6. Alternate form of Euler equation.

Returning to the case in which the integrand is a function of only one generalized coordinate, $\mathcal{L}$ can vary with time both explicitly and also through the time dependence of y or $\dot{y}$:

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial t} + \frac{\partial \mathcal{L}}{\partial y}\dot{y} + \frac{\partial \mathcal{L}}{\partial \dot{y}}\ddot{y}.$$

Using the Euler equation (4.3) for $\partial\mathcal{L}/\partial y$,

$$\begin{aligned} \frac{d\mathcal{L}}{dt} &= \frac{\partial \mathcal{L}}{\partial t} + \dot{y}\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} + \frac{\partial \mathcal{L}}{\partial \dot{y}}\ddot{y} \\ &= \frac{\partial \mathcal{L}}{\partial t} + \frac{d}{dt}(\dot{y}\frac{\partial \mathcal{L}}{\partial \dot{y}}) \\ -\frac{\partial \mathcal{L}}{\partial t} &= \frac{d}{dt}(\dot{y}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \mathcal{L}) \equiv \frac{d}{dt}\mathcal{H}. \end{aligned} \tag{4.10}$$

If $\mathcal{L}$ is free of *explicit* time dependence, $\mathcal{H} = E$, where E is a constant. This equality can substitute for the Euler equation in solving the same extremization problem. It leads to much simpler algebra in some cases.

If $\mathcal{L}$ is the Lagrangian $T - U$, and if t is the time, $\mathcal{H}$ as defined in (4.10) is the *Hamiltonian*. If $\mathcal{L}$ is free from explicit time dependence, the constant E is the conserved total mechanical energy. If U is independent of $\dot{y}$ (velocity-independent potential) and if T is a quadratic function of $\dot{y}$,

$$\begin{aligned} E = \mathcal{H} &\equiv \dot{y}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \mathcal{L} \\ &= 2T - (T - U) = T + U. \end{aligned}$$

Under these conditions, the conserved total mechanical energy is the sum of kinetic and potential energies as expected.

5. Lagrangian mechanics.

5.1. Hamilton's principle.

The foregoing discussion of the calculus of variations, with or without equations of constraint, acquires physical relevance from a famous postulate by Hamilton. From now on, the independent variable t will represent the time, and the coordinates y_i will be called *generalized coordinates*. They are "generalized" in the sense that any time-dependent quantity that helps to define the state of a system (Cartesian coordinate, spherical or cylindrical coordinate, Euler angle, etc.) can be chosen as a generalized coordinate. Not all the y_i ($1 \le i \le n$) in the same problem need to have the same dimension. The path $y_i(t)$ followed by a system is called its *history*.

For systems in which all the forces are conservative, Hamilton's principle states that the history which the system actually will follow is that which makes the action J *stationary*, where

$$J \equiv \int_{t_1}^{t_2} \mathcal{L}(y_i, \dot{y}_i, t)\, dt. \tag{5.1}$$

Here $\mathcal{L}$ is the Lagrangian $T - U$. The calculus of variations supplies n Euler-Lagrange equations that may be solved for the history:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}_i} = \frac{\partial \mathcal{L}}{\partial y_i}. \tag{5.2}$$

As another choice, we may define the Hamiltonian, and, if $\partial\mathcal{L}/\partial t = 0$, we may use the alternate equation

$$\begin{aligned} \mathcal{H}(y_i, \dot{y}_i) &\equiv \dot{y}_i \frac{\partial \mathcal{L}}{\partial \dot{y}_i} - \mathcal{L} \\ &= E = \text{constant}. \end{aligned} \tag{5.3}$$

We accept Hamilton's principle because it is found to reproduce all solutions obtained using Newtonian analysis, and because it agrees with experimental observation for additional classes of problem beyond the reach of Newtonian analysis. Hamiltonian analysis is also the historical path to quantum mechanics.

Often T and U are generalized quadratic functions of the y_i and $\dot{y}_i$ respectively:

$$T = \tfrac{1}{2} T_{ij} \dot{y}_i \dot{y}_j \quad U = \tfrac{1}{2} U_{ij} y_i y_j, \tag{5.4}$$

where the T_{ij} are functions only of the y_i, and the U_{ij} are functions only of the $\dot{y}_i$. In this case $\mathcal{H} = T + U$, and the Euler-Lagrange equations simplify to

$$T_{ij} \ddot{y}_j = -U_{ij} y_j. \tag{5.5}$$

If, for some i, $T_{ij} = m\delta_{ij}$ and $U_{ij} = k\delta_{ij}$, the Euler-Lagrange equation for the i^{th} Cartesian coordinate looks like a component of $m\mathbf{a} = \mathbf{F}$:

$$m\ddot{y}_i = -k y_i.$$

Evidently $(d/dt)(\partial\mathcal{L}/\partial\dot{y}_i)$ plays the role of the rate of change of a momentum, and $\partial\mathcal{L}/\partial y_i$ plays the role of a force. More generally, $(\partial\mathcal{L}/\partial\dot{y}_i)$ is called a *generalized momentum* and $(\partial\mathcal{L}/\partial y_i)$ is called a *generalized force*. For example, if y_i is really an angle, the corresponding generalized momentum is really an angular momentum, and the generalized force is really a torque.

Equation (4.9) added Lagrange multiplier terms to the Euler-Lagrange equations for the case in which two generalized coordinates were mutually constrained. More generally, there may be n generalized coordinates y_i and m constraint equations, $1 \le j \le m$:

$$g_{ji} dy_i + g_{jt} dt = 0. \tag{5.6}$$

Each of the $m \times (n+1)$ g's is a function of the $2n+1$ variables $(y_i, \dot{y}_i, t)$. The problem is solved by adding n Euler-Lagrange equations each containing m Lagrange multipliers λ_j:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}_i} = \frac{\partial \mathcal{L}}{\partial y_i} + g_{ji}\lambda_j. \tag{5.7}$$

In addition to the generalized force $\partial\mathcal{L}/\partial y_i$ that is derived from the potential U, the right-hand side contains an additional term $g_{ji}\lambda_j$ (summation implied) which is the *generalized force of constraint* Q_i. This is the physical significance of the Lagrange multipliers: when we solve for the λ's as well as the y_i's, we obtain the forces of constraint as well as the motion. To summarize, $n+m$ unknown functions $y_i(t)$ and $\lambda_j(t)$ are solved by n Euler-Lagrange equations containing Lagrange multipliers, plus m equations of constraint.

Note that the generalized forces of constraint *must do no work* – otherwise we could no longer continue to define the potential energy U. For example, the force of constraint from a wall must be normal rather than frictional.

5.2. The falling ladder.

This interesting problem is solved straightforwardly by using the Euler-Lagrange equations with a Lagrange multiplier. At $t = 0$ a ladder of length h under the influence of gravity is released from rest. The bottom is in contact with a frictionless floor $y = 0$. The top rests on a frictionless wall $x = 0$ with which the ladder makes an initial angle α_0.

Part (a) of the problem is just a warm-up: Assuming $\alpha \ll 1$, find $\alpha(t)$.

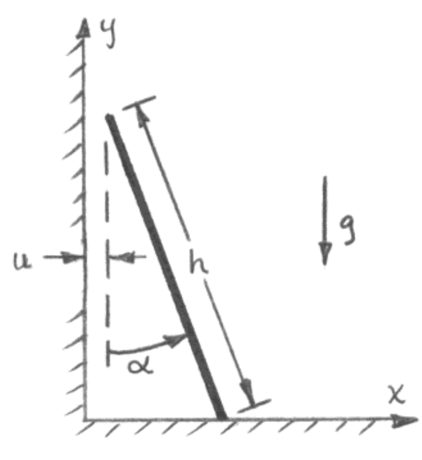
For this part we assume that the ladder remains in contact with the wall. Then its position is specified by only one generalized coordinate, which we take as α. Then the C.M. of the ladder is at $x = (h/2)\sin\alpha$, $y = (h/2)\cos\alpha$. The Lagrangian is:

$$
\begin{aligned}
\mathcal{L} &= T - U \\
&= T_{\text{trans}} + T_{\text{rot}} - U \\
&= \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 - U \\
&= \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}(\tfrac{1}{12}mh^2\dot{\alpha}^2) - \tfrac{1}{2}mgh\cos\alpha \\
&= \tfrac{1}{8}mh^2\dot{\alpha}^2 + \tfrac{1}{24}mh^2\dot{\alpha}^2 - \tfrac{1}{2}mgh\cos\alpha \\
&= \tfrac{1}{6}mh^2\dot{\alpha}^2 - \tfrac{1}{2}mgh\cos\alpha.
\end{aligned}
$$

Applying the Euler-Lagrange equation (5.2),

$$
\begin{aligned}
\frac{d}{dt}\tfrac{1}{3}mh^2\dot{\alpha} &= \tfrac{1}{2}mgh\sin\alpha \\
\tfrac{1}{3}mh^2\ddot{\alpha} &= \tfrac{1}{2}mgh\sin\alpha \\
\ddot{\alpha} &= \frac{3g}{2h}\sin\alpha \approx \frac{3g}{2h}\alpha \\
\alpha &\approx \alpha_0 \cosh\sqrt{\frac{3g}{2h}}t,
\end{aligned}
$$

where in the last equation the boundary conditions have been applied.



Part (b) of the falling ladder problem is more interesting: *Without* assuming $\alpha \ll 1$, find the angle α_1 at which the ladder leaves the wall.

We relax the requirement that the top of the ladder rest against the wall by introducing a second coordinate u, the distance between the top of the ladder and the wall. The equation of constraint imposed by the wall then is $u \geq 0$. Since inequalities are difficult to handle, we instead impose the constraint $u = 0$. Using the method of Lagrange multipliers, we solve for the generalized force of constraint $Q_u = \lambda g_u$. The ladder will leave the wall when the wall exerts no force on it, i.e. when $Q_u = 0$.

Re-expressed in terms of α and u, the C.M. coordinate y is the same, but x acquires the extra term u. This leads to two extra terms

$$\tfrac{1}{2}mh\cos\alpha\dot{\alpha}\dot{u} + \tfrac{1}{2}m\dot{u}^2$$

in T. The constraint equation is

$$
\begin{gathered}
g_\alpha d\alpha + g_u du + g_t dt = 0 \\
g_\alpha = 0 \quad g_u = 1 \quad g_t = 0.
\end{gathered}
$$

When we include the Lagrange multiplier, the Euler-Lagrange equation in α is unchanged, since $\dot{u} = 0$ and $g_\alpha = 0$. The equation in u is

$$
\begin{aligned}
\frac{d}{dt}\left[\tfrac{1}{2}m(h\cos\alpha\dot{\alpha} + 2\dot{u})\right] &= \lambda(t) \\
-\sin\alpha\dot{\alpha}^2 + \cos\alpha\ddot{\alpha} &= \frac{2}{mh}\lambda(t).
\end{aligned}
$$

In the second equation, we used the fact that $\dot{u} = 0$. When the ladder leaves the wall, $\alpha \equiv \alpha_1$, and $\lambda = 0$. The Euler-Lagrange equation in u becomes

$$\ddot{\alpha}\big|_{\alpha_1} = \dot{\alpha}^2\tan\alpha\big|_{\alpha_1}.$$

Substituting for $\ddot{\alpha}$ from the Euler-Lagrange equation in α,

$$\frac{3g}{2h}\cos\alpha\big|_{\alpha_1} = \dot{\alpha}^2\big|_{\alpha_1}. \tag{5.8}$$

Equation (5.8) would solve the problem if we could obtain a condition expressing $\dot{\alpha}$ in terms of α. Then we would have an equation for α_1 that might be solved. Such a condition is provided by the alternate form of the Euler-Lagrange equation. Returning to the analysis in terms of a single generalized coordinate α, appropriate for the part of the motion in which the ladder remains in contact with the wall, we define

$$\begin{aligned}\mathcal{H} &\equiv \dot{\alpha}\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} - \mathcal{L} \\ &= \dot{\alpha}\tfrac{1}{3}mh^2\dot{\alpha} - \tfrac{1}{6}mh^2\dot{\alpha}^2 + \tfrac{1}{2}mgh\cos\alpha \\ &= \tfrac{1}{6}mh^2\dot{\alpha}^2 + \tfrac{1}{2}mgh\cos\alpha = E,\end{aligned}$$

where in the last equation we used the fact that $\mathcal{L}$ does not depend explicitly on the time. Equating $\mathcal{H}$ above to $\mathcal{H}$ at $(t = 0, \alpha = \alpha_0, \dot{\alpha} = 0)$,

$$\tfrac{1}{6}mh^2\dot{\alpha}^2 + \tfrac{1}{2}mgh\cos\alpha = \tfrac{1}{2}mgh\cos\alpha_0.$$

Setting $\alpha = \alpha_1$ and substituting Eq. (5.8) for $\dot{\alpha}$,

$$\begin{aligned}\tfrac{1}{6}mh^2\frac{3g}{2h}\cos\alpha_1 &= \tfrac{1}{2}mgh(\cos\alpha_0 - \cos\alpha_1) \\ \tfrac{3}{2}\cos\alpha_1 &= \cos\alpha_0 \\ \alpha_1 &= \cos^{-1}(\tfrac{2}{3}\cos\alpha_0).\end{aligned}$$

If the ladder starts at a certain cosine, it leaves the wall when the height of the ladder decreases to $\frac{2}{3}$ of its initial value. For example, if it starts upright (at 0°), it leaves at $\cos^{-1}\frac{2}{3} = 48.2°$.

5.3. Cyclic coordinates and conservation laws.

In section 5.1 we identified $\partial\mathcal{L}/\partial\dot{q}_i$ as a generalized momentum (we have substituted $\dot{q}_i$ for $\dot{y}_i$, following the notation traditionally used for this topic). In fact, this generalized momentum is given a longer name – the *canonical momentum conjugate to* q_i, or (more succinctly) p_i. The reason for the emphasis is that, when there is no constraint equation and $\partial\mathcal{L}/\partial q_i$ vanishes, the Euler-Lagrange equation requires that the canonical momentum be *conserved*. $\mathcal{L}$ is said to be *cyclic* in q_i.

If a system is closed, the homogeneity of spacetime requires the Lagrangian to be invariant to displacements or rotations of the coordinate system, and also to displacements in the zero of time. Using the canonical momenta, each of these invariance principles gives rise to a conservation law.

Consider a displacement δ_x in all the linear coordinates q_x^j in the x direction of the various particles j. The Lagrangian is unchanged under this displacement:

$$\begin{aligned}0 = \frac{\partial \mathcal{L}}{\partial \delta_x} &= \sum_j \frac{\partial \mathcal{L}}{\partial q_x^j}\frac{\partial q_x^j}{\partial \delta_x} \\ &= \sum_j \frac{\partial \mathcal{L}}{\partial q_x^j} \\ &= \frac{d}{dt}\sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_x^j} \\ &\equiv \frac{d}{dt}\sum_j p_x^j \equiv \frac{d}{dt}P_x.\end{aligned} \tag{5.9}$$

(The first equation does *not* imply summation over x, and the third equation uses the fact that $\partial q_x^j/\partial\delta_x = 1$). Thus P_x, the total linear momentum in the x direction, is conserved.

Consider next a rotation ϵ_x about the x axis. This is equivalent to a displacement ϵ_x in all the angular coordinates θ_x^j about the x axis of the various particles j. By a similar derivation, invariance of the Lagrangian to this rotation leads to the conservation of total angular momentum L_x:

$$\begin{aligned}0 &= \frac{d}{dt}\sum_j \frac{\partial \mathcal{L}}{\partial \dot{\theta}_x^j} \\ &\equiv \frac{d}{dt}\sum_j l_x^j \equiv \frac{d}{dt}L_x.\end{aligned} \tag{5.10}$$

Finally, we saw in Eq. (5.3) that invariance of $\mathcal{L}$ to a displacement in time t causes the Hamiltonian $\mathcal{H}$ to be conserved. Thus $\mathcal{H}$ must bear a canonical relationship to t similar to that of P_x to δ_x or of L_x to ϵ_x. However, the analogy is not complete, because t plays a special role as the independent variable upon which the q's and p's depend.

6. Hamiltonian mechanics.

6.1. Hamilton's equations.

We have already introduced one equation involving the Hamiltonian $\mathcal{H}$. Equation (4.10) defined $\mathcal{H}$ and obtained its time derivative:

$$\frac{d\mathcal{H}(q_i, \dot{q}_i, t)}{dt} = -\frac{\partial \mathcal{L}(q_i, \dot{q}_i, t)}{\partial t}. \tag{6.1}$$

In Eq. (6.1), the Hamiltonian is (temporarily!) considered to be a function of the same variables as is the Lagrangian.

However, we are advised to put Eq. (6.1) on the back burner, because $\mathcal{H}$ instead is normally considered to be a function of the generalized coordinates, their *canonically conjugate momenta*, and the time: $\mathcal{H} = \mathcal{H}(q_i, p_i, t)$. When expressed in terms of the p_i rather than the $\dot{q}_i$, the Hamiltonian may take on a different functional form. For example, using polar coordinates r and θ, a free particle has the Hamiltonia

$$\begin{aligned} \mathcal{H}(r, \theta, \dot{r}, \dot{\theta}, t) &= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \\ \mathcal{H}(r, \theta, p_r, p_\theta, t) &= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right). \end{aligned}$$

Considering $\mathcal{H}$ to be a function of (q_i, p_i, t), and $\mathcal{L}$ still to be a function of $(q_i, \dot{q}_i, t)$, we examine the total differential of $\mathcal{H}$:

$$\begin{aligned} \mathcal{H} &= \dot{q}_i p_i - \mathcal{L} \\ d\mathcal{H} &= \dot{q}_i dp_i + p_i d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial q_i} dq_i - \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial t} dt \\ &= \dot{q}_i dp_i + p_i d\dot{q}_i - \dot{p}_i dq_i - p_i d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial t} dt \\ &= \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial \mathcal{L}}{\partial t} dt. \end{aligned} \tag{6.2}$$

(In the third equality we made use of the Euler-Lagrange equations.)

On the other hand, $\mathcal{H}$ formally is a function of (q_i, p_i, t):

$$d\mathcal{H} \equiv \frac{\partial \mathcal{H}}{\partial q_i} dq_i + \frac{\partial \mathcal{H}}{\partial p_i} dp_i + \frac{\partial \mathcal{H}}{\partial t} dt. \tag{6.3}$$

Since $\mathcal{H}$ depends independently on q_i, p_i, and t, Eqs. (6.2) and (6.3) must be equivalent term by term. The result is *Hamilton's equations*:

$$\begin{aligned} \mathcal{H} &= \mathcal{H}(q_i, p_i, t) \\ \dot{q}_i &= +\frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial \mathcal{H}}{\partial q_i} \\ \frac{\partial \mathcal{H}}{\partial t} &= -\frac{\partial \mathcal{L}}{\partial t}. \end{aligned} \tag{6.4}$$

Note that the total and partial time derivatives of $\mathcal{H}(q_i, p_i, t)$ (but not $\mathcal{H}(q_i, \dot{q}_i, t)$!) are equivalent:

$$\begin{aligned} \frac{d\mathcal{H}}{dt} &= \frac{\partial \mathcal{H}}{\partial q_i}\dot{q}_i + \frac{\partial \mathcal{H}}{\partial p_i}\dot{p}_i + \frac{\partial \mathcal{H}}{\partial t} \\ &= -\dot{p}_i\dot{q}_i + \dot{q}_i\dot{p}_i + \frac{\partial \mathcal{H}}{\partial t} \\ &= \frac{\partial \mathcal{H}}{\partial t}. \end{aligned}$$

Therefore, the last of Hamilton's equations is written with equal validity using $d\mathcal{H}/dt$ or $\partial\mathcal{H}/\partial t$. In contrast, Eq. (6.1) considered $\mathcal{H}$ still to be a function of $(q_i, \dot{q}_i, t)$; we do not substitute the partial time derivative of $\mathcal{H}$ there.

6.2. The Poisson bracket.

Applied to straightforward problems that are amenable to Lagrangian analysis, Hamilton's equations substitute two coupled partial differential equations of first order for one Euler-Lagrange equation – an ordinary differential equation of second order. Usually, this is no bargain. Most often, after manipulation, one obtains from Hamiltonian analysis the same differential equations that are found more easily from the Lagrangian.

For more difficult problems, there exists a prescription for making a *canonical transformation* to a new set of generalized coordinates Q_i and canonically conjugate momenta P_i, so that (at least) all the P_i are constants of the motion. Although Hamilton's equations become trivial when expressed in terms of the new variables, finding the right canonical transformation requires solving partial differential equations that are not trivial. This is the most powerful method for analyzing problems in classical mechanics. It is beyond the scope of a one-semester undergraduate course.

For present purposes, the attraction of Hamilton's equations is that they encourage us to think democratically about coordinates and momenta, and that they reveal essential ideas which led from classical to quantum mechanics. The *Poisson bracket* is a good example of both attractions.

Consider any quantity $\rho(q_i, p_i, t)$ that is a function of the same variables as $\mathcal{H}$. Its total time derivative is

$$\begin{aligned} \frac{d\rho}{dt} &= \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial q_i}\dot{q}_i + \frac{\partial\rho}{\partial p_i}\dot{p}_i \\ &= \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial q_i}\frac{\partial\mathcal{H}}{\partial p_i} - \frac{\partial\rho}{\partial p_i}\frac{\partial\mathcal{H}}{\partial q_i} \\ &\equiv \frac{\partial\rho}{\partial t} + [\rho, \mathcal{H}]. \end{aligned} \tag{6.5}$$

In the second equality we used Hamilton's equations, and in the last we defined the *Poisson bracket* of ρ and $\mathcal{H}$,

$$[\rho, \mathcal{H}] \equiv \frac{\partial\rho}{\partial q_i}\frac{\partial\mathcal{H}}{\partial p_i} - \frac{\partial\rho}{\partial p_i}\frac{\partial\mathcal{H}}{\partial q_i}, \tag{6.6}$$

as usual with summation over i implied. Equation (6.5) states that the *implicit* time derivative of any function of (q_i, p_i, t) is given by the Poisson bracket of that function with $\mathcal{H}$: the Hamiltonian is the unique function which controls the time evolution of all other functions. (It is even easier to see why $d\mathcal{H}/dt = \partial\mathcal{H}/\partial t$, as noted in the previous section: the Poisson bracket of $\mathcal{H}$ with itself obviously vanishes.)

In quantum mechanics, if ρ were an *operator*, one would write

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{1}{i\hbar}[\rho, \mathcal{H}],$$

where $[\rho, \mathcal{H}] \equiv \rho\mathcal{H} - \mathcal{H}\rho$ is the *commutator* of ρ with the Hamiltonian operator. The quantity $\hbar$, with the same dimension as pq, is *Planck's constant* h divided by 2π. Thus the Poisson bracket is the quantity in classical mechanics that led to the quantum mechanical commutator.

6.3. Phase space.

Consider a system of n particles in three-dimensional space. This system is described by a set of $6n$ variables $\{q_i, p_i\}$, where, for example, $i = 1\,2\,3$ for $x\,y\,z$ of the first particle, $4\,5\,6$ for $x\,y\,z$ of the second particle, etc. The system is represented as a *single point* in a $6n$ dimensional space called *phase space*.

If we are required to consider an *ensemble* of a large number N of identical systems, each satisfying initial conditions that (in principle) could be unique, the number $(3nN)$ of Euler-Lagrange equations becomes unwieldy. To learn about the average behavior of these systems, we are motivated to consider the *statistical* properties of the ensemble. Suppose that we prepare an ensemble of N systems initially within a closed boundary in $6n$-dimensional phase space. The ensemble can be visualized as a N points within a $6n$-dimensional bag. After a time, each system evolves to a new point in phase space. The bag also evolves to a new shape.

Our first conclusion is that any system that initially lies within the closed boundary will always lie within the (evolving) boundary: *a bounded system remains bounded.* The reason is simple: if at some time a system lies on a phase space point which is part of the boundary, from that time forward it must evolve *with the boundary.* It cannot cross the boundary. Otherwise, we would have two identical systems (one remaining on the boundary, one crossing it) that evolve differently given the same initial conditions.

6.4. Liouville's theorem.

This celebrated theorem has wide-ranging practical implications. It states that the volume in phase space occupied by an ensemble of systems remains constant as the systems evolve, when the Hamiltonian is constant or even when it varies (smoothly) with time. Let ρ be the number of systems per unit phase space volume. (As an aside, if each system were a spin $\frac{1}{2}$ fermion such as an electron, which obeys the

Pauli exclusion principle, the maximum value of ρ would be $2h^{-3n}$, where the factor of 2 allows the electron spin to point up or down.)

To analyze the evolution of ρ we need the $6n$-dimensional velocity $\mathbf{v}_p$ and the $6n$-dimensional gradient operator ∇_p:

$$\begin{aligned} \mathbf{v}_p &\equiv \hat{q}_i \dot{q}_i + \hat{p}_i \dot{p}_i \\ \nabla_p &\equiv \hat{q}_i \frac{\partial}{\partial q_i} + \hat{p}_i \frac{\partial}{\partial p_i}, \end{aligned}$$

where summation over i is implied as usual. In the $6n$-dimensonal space, the number N of systems is conserved: an increase in the density of systems in a certain region requires a net flux of systems into the region. This is expressed quantitatively by an equation of continuity:

$$\frac{\partial \rho}{\partial t} = -\nabla_p \cdot (\rho \mathbf{v}_p). \tag{6.7}$$

Using Eq. (6.5), the total time derivative of ρ is

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + [\rho, \mathcal{H}].$$

Substituting for $\partial\rho/\partial t$ from Eq. (6.7),

$$\begin{aligned} \frac{d\rho}{dt} &= [\rho, \mathcal{H}] - \nabla_p \cdot (\rho v_p) \\ &= [\rho, \mathcal{H}] - \{\frac{\partial}{\partial q_i}(\rho \dot{q}_i) + \frac{\partial}{\partial p_i}(\rho \dot{p}_i)\}. \end{aligned} \tag{6.8}$$

Using Hamilton's equations for $\dot{q}_i$ and $\dot{p}_i$, the curly bracket in Eq. (6.8) is

$$\frac{\partial \rho}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial \rho}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i} + \rho[\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i}].$$

The first two terms are just $[\rho, \mathcal{H}]$, cancelling the Poisson bracket in Eq. (6.8). The terms in the square bracket are equal, respectively, to

$$+\frac{\partial^2 \mathcal{H}}{\partial q_i \partial p_i} \quad \text{and} \quad -\frac{\partial^2 \mathcal{H}}{\partial p_i \partial q_i}.$$

These terms also cancel, provided that $\mathcal{H}$ varies smoothly with q_i and p_i, so that the order of differentiation does not matter.

We have proven Liouville's theorem,

$$\frac{d\rho}{dt} = 0.$$

Since the phase space density ρ and the number of systems N are constant, the volume N/ρ in phase space must also be constant.

It is significant that the phase space volume is proportional to the range in a particular generalized coordinate q_j multiplied by the range in its canonically conjugate momentum p_j. Suppose, at a certain time, that an ensemble of systems occupies a range Δq_j and a range Δp_j. Later, suppose that Δq_j *decreases* by a factor, and there is no change in the range of other coordinates q_i and momenta p_i with $i \neq j$. Then Δp_j must *increase* by the same factor, so that

$$\Delta q_j \Delta p_j = \text{constant}. \tag{6.9}$$

Nothing in classical mechanics suggests that Δq_j or Δp_j should be interpreted as an *uncertainty*. In the present context, each is merely the range of values accessible to an ensemble of identical systems satisfying different initial conditions. But if, for the moment, we *were* to entertain such an interpretation, Eq. (6.9) would resemble an *uncertainty principle* of the type Heisenberg introduced to quantum mechanics. We would be led to conclude that an uncertainty principle can be written for the pair of quantum mechanical operators corresponding to *any* pair of canonically conjugate classical variables. This turns out to be the case.

As a simple application of Liouville's theorem, imagine a truncated cone of half-angle $\ll 1$ that is perfectly reflecting on the inside. Randomly directed, nearly monochromatic photons enter the larger end. Naively, one might hope that the photons would all be funneled down the cone through the small end, yielding an intensified spot of light.

Liouville would reach a different conclusion. At the larger end (area A), the photons entering the cone already occupy a volume in momentum space equal to that of a hemispherical shell with a thickness corresponding to the small range in photon |momentum|. At the smaller end (area a), the emerging photons would occupy this same volume in momentum space. Since the volume in position ("configuration") space is a/A times smaller there, and the phase space density must remain constant, only a fraction a/A of the photons can emerge from the small end; the remainder must be multiply reflected back to the large end of the cone. (The condition that $\mathcal{H}$ must vary smoothly with the q_i and p_i is equivalent, in this problem, to the requirement that the cone aperture may change significantly only over a distance along the cone corresponding typically to at least a few photon reflections.)

7. Central force motion.

7.1. Reduced mass.

We consider the central force problem with two point masses m_1 and m_2 separated by a vector $\mathbf{r}$ pointing from m_1 to m_2. From the C.M., vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ point to each mass respectively, so that $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. The C.M. is defined by $m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0$.

In principle, the two-body system requires six coordinates to describe it – three for the C.M. coordinate $\mathbf{R}$, and three e.g. for $\mathbf{r}$. We want to reduce this number in order to simplify the analysis. By working in the C.M. system, we eliminate $\mathbf{R}$ from further consideration. About the C.M., either of the (parallel) angular momenta $\mathbf{L}_1 \equiv m_1\mathbf{r}_1\times\dot{\mathbf{r}}_1$ and $\mathbf{L}_2 \equiv m_2\mathbf{r}_2\times\dot{\mathbf{r}}_2$ defines the normal to the plane of relative motion. Since the force between the bodies is central, it can exert no torque about the C.M., so that $\mathbf{L}_1+\mathbf{L}_2$, and therefore the plane of relative motion, must remain invariant. As our two generalized coordinates, we choose r and θ, the azimuthal angle of $\mathbf{r}$ in the plane of relative motion.

In the C.M., the kinetic energy is

$$T = \tfrac{1}{2}m_1\dot{r}_1^2 + \tfrac{1}{2}m_1 r_1^2\dot{\theta}^2 + \tfrac{1}{2}m_2\dot{r}_2^2 + \tfrac{1}{2}m_2 r_2^2\dot{\theta}^2.$$

Substituting for r_1 and r_2 in terms of r, one finds easily

$$\begin{aligned} T &= \tfrac{1}{2}\mu\dot{r}^2 + \tfrac{1}{2}\mu r^2\dot{\theta}^2, \\ \mu &\equiv \frac{m_1 m_2}{m_1+m_2}. \end{aligned} \tag{7.1}$$

The quantity μ last defined is the *reduced mass.* It is always between 50% and 100% of the *smaller* of m_1 and m_2.

7.2. Constants of the motion.

Expressed in terms of the reduced mass, the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r), \tag{7.2}$$

and the Hamiltonian is

$$\begin{aligned} \mathcal{H} &= \dot{r}\frac{\partial\mathcal{L}}{\partial\dot{r}} + \dot{\theta}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \mathcal{L} \\ &= 2T - (T-U) = T + U \\ &= \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r). \end{aligned}$$

In terms of the canonical momenta,

$$\begin{aligned} p_r &\equiv \frac{\partial\mathcal{L}}{\partial\dot{r}} = \mu\dot{r} \\ p_\theta &\equiv \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \mu r^2\dot{\theta}, \end{aligned}$$

the Hamiltonian is more naturally written

$$\mathcal{H} = \frac{p_r^2}{2\mu} + \frac{p_\theta^2}{2\mu r^2} + U(r).$$

Because the θ coordinate is cyclic ($\partial\mathcal{L}/\partial\theta = 0$), the Euler-Lagrange equation requires the canonically conjugate momentum p_θ (in this case an *angular* momentum) to be conserved:

$$p_\theta = \text{constant} \equiv l. \tag{7.3}$$

Because $\partial\mathcal{L}/\partial t = 0$, the Hamiltonian is constant as well:

$$\begin{aligned} \mathcal{H} &= \text{constant} \equiv E \\ E &= \frac{p_r^2}{2\mu} + \frac{l^2}{2\mu r^2} + U(r) \\ E &= \frac{\mu\dot{r}^2}{2} + \frac{l^2}{2\mu r^2} + U(r). \end{aligned} \tag{7.4}$$

This last equation involving the single coordinate r is the basis of the analysis that follows.

7.3. The repulsive pseudopotential.

We know that the term $l^2/2\mu r^2$ on the right-hand side of Eq. (7.4) is part of the kinetic energy. However, because it depends on r rather than $\dot{r}$, it has the functional form of a *potential* energy, and is called the *pseudopotential* U'. Near the origin, U' is singular and positive, and therefore repulsive.

If the actual potential $U(r)$ is attractive (negative), and is proportional to r^{n+1} with $-3 < n < -1$, the sum of U and U' is $+\infty$ at $r = 0$ and 0 at $r = \infty$, falling to a minimum $-|U_0|$ at some finite radius r_0. If E is negative, the pair of masses is a *bound system*. If $E = -|U_0|$, all of the kinetic energy arises from angular motion $l^2/2\mu r_0^2$ rather than radial motion $\frac{1}{2}\mu\dot{r}^2$, and the masses are in a *stable circular orbit*. If E is larger than $-|U_0|$, but still negative, the radius of the orbit varies between the perigee $r_{\min}$ and the apogee $r_{\max}$. At each of these classical turning points, $E = U + U'$. When $n = -2$, corresponding to a gravitational or Coulomb potential, the orbit is an *ellipse* when $-|U_0| < E < 0$, a *parabola* when $E = 0$, or a *hyperbola* when $E > 0$.

7.4. Period and orbit shape.

Defining

$$\mathcal{E} \equiv \frac{2\mu E}{l^2};\;\; \mathcal{U} \equiv \frac{2\mu U}{l^2},$$

Eq. (7.4) becomes

$$\begin{aligned}\mathcal{E} &= (\frac{\mu\dot{r}}{l})^2 + r^{-2} + \mathcal{U}(r)\\ \frac{l}{\mu}\frac{dt}{dr} &= (\mathcal{E} - \mathcal{U}(r) - r^{-2})^{-1/2}\\ \frac{\mathcal{T}}{2} &\equiv \int_{r_{\min}}^{r_{\max}} dt\\ &= \frac{\mu}{l}\int_{r_{\min}}^{r_{\max}} \frac{dr}{\sqrt{\mathcal{E} - \mathcal{U}(r) - r^{-2}}},\end{aligned} \tag{7.5}$$

where $\mathcal{T}$ is the period of radial oscillation, not necessarily equal to the gross orbital period even if the orbit is closed.

Alternatively, setting

$$\frac{d\theta}{dr} = \frac{d\theta}{dt}\frac{dt}{dr} = \frac{l}{\mu r^2}\frac{dt}{dr},$$

a similar integral gives the orbit shape $\theta(r)$:

$$\begin{aligned}\theta_2 - \theta_1 &\equiv \int_{r_1}^{r_2} d\theta\\ &= \int_{r_1}^{r_2} \frac{dr/r^2}{\sqrt{\mathcal{E} - \mathcal{U}(r) - r^{-2}}}.\end{aligned} \tag{7.6}$$

If $\mathcal{U}(r) = kr^{n+1}$, with $n = +5, +3, 0, -4, -5$, or -7, this is an *elliptic integral* found in tables. More importantly, numerical integration can yield an arbitrarily precise orbit for any well-behaved potential.

Conversely, substituting $u \equiv 1/r$, Eq. (7.6) can be re-expressed as

$$\begin{aligned}-d\theta &= (\mathcal{E} - \mathcal{U}(u) - u^2)^{-1/2} du\\ (\frac{du}{d\theta})^2 &= \mathcal{E} - \mathcal{U}(u) - u^2,\end{aligned} \tag{7.7}$$

yielding the potential $\mathcal{U}(u)$, and thus the force law, given the orbit shape $u(\theta)$.

7.5. Orbit for inverse square law force.

For the planetary case ($U = -k/r$), the orbit is easily expressed in closed form. Equation (7.7) becomes

$$(\frac{du}{d\theta})^2 = \mathcal{E} + \frac{2\mu k u}{l^2} - u^2.$$

Differentiating with respect to θ and dividing through by $du/d\theta \neq 0$ (noncircular motion), we obtain

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu k}{l^2}. \tag{7.8}$$

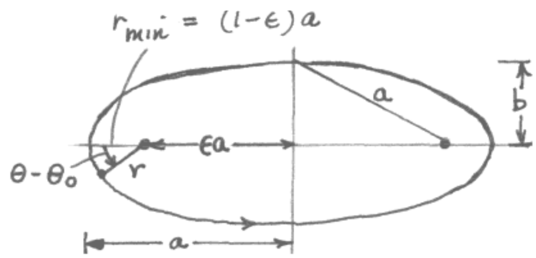
The particular and homogeneous solutions are, respectively,

$$\begin{aligned}u_p &= \frac{\mu k}{l^2}\\ u_h(\theta) &= \frac{\mu k}{l^2}\epsilon\cos(\theta - \theta_0),\end{aligned}$$

where $\dot{u}(\theta_0) \equiv 0$, and $\mu k\epsilon/l^2$ is an adjustable constant. Combining u_p and u_h,

$$u(\theta) \equiv \frac{1}{r(\theta)} = \frac{\mu k}{l^2}(1 + \epsilon\cos(\theta - \theta_0)). \tag{7.9}$$

When $0 < \epsilon < 1$, this is the equation of an *ellipse* with *eccentricity* ϵ. The origin about which θ is measured is one of its two *foci*. These are located within the ellipse, but are well separated from its center if ϵ is significantly greater than zero, i.e. if the ellipse is eccentric.



7.6. Kepler's laws.

We have already proven two of Kepler's laws. The first is that the *areal velocity*,

$$\frac{dA}{dt} = \frac{|\mathbf{r} \times d\mathbf{r}|}{2\,dt} = \frac{l}{2\mu},$$

where dA is an increment of orbit area, is constant. This follows directly from conservation of angular momentum l. The second of Kepler's laws states that the orbits are ellipses with foci at the origin, as shown in Eq. (7.9).

Our remaining objectives are to prove that the total energy is a function only of the major axis of the orbit ellipse, and to establish the third and last of Kepler's laws, which relates the orbit period to the major axis. To proceed, we evaluate Eq. (7.9) at $\theta = \theta_0$ (perigee r_{min}), and at $\theta = \theta_0 + \pi$ (apogee r_{max}):

$$\begin{aligned} r_{\mathrm{min}} &= \frac{l^2}{\mu k(1+\epsilon)}; \quad r_{\mathrm{max}} = \frac{l^2}{\mu k(1-\epsilon)} \\ a &\equiv \frac{r_{\mathrm{min}} + r_{\mathrm{max}}}{2} = \frac{l^2}{\mu k(1-\epsilon^2)}, \end{aligned} \tag{7.10}$$

where a is the *semimajor axis*. In terms of a, we can rewrite

$$r_{\mathrm{min}} = a(1-\epsilon); \quad r_{\mathrm{max}} = a(1+\epsilon). \tag{7.11}$$

At the perigee, where $\dot{r} = 0$, the total energy is

$$\begin{aligned} E &= \frac{l^2}{2\mu r_{\mathrm{min}}^2} - \frac{k}{r_{\mathrm{min}}} \\ &= \frac{l^2}{2\mu a^2(1-\epsilon)^2} - \frac{k}{a(1-\epsilon)} \\ &= \frac{ka(1-\epsilon^2)}{2a^2(1-\epsilon)^2} - \frac{k}{a(1-\epsilon)} \\ &= \frac{k(1+\epsilon)}{2a(1-\epsilon)} - \frac{k}{a(1-\epsilon)} \\ &= \frac{k}{a(1-\epsilon)}\Big(\frac{1+\epsilon}{2} - 1\Big) \\ &= -\frac{k}{2a}. \end{aligned} \tag{7.12}$$

(In the third line we substituted $l^2/\mu = ka(1-\epsilon^2)$ from Eq. (7.10).) Therefore the total energy is a function only of the semimajor axis.

To find the period, it is straightforward to integrate Eq. (7.5). The same result is obtained with less algebra by considering the orbit area A:

$$\mathcal{T} = \frac{A}{dA/dt} = \frac{\pi ab}{l/2\mu}, \tag{7.13}$$

where b is the *semiminor axis*. To evaluate b, consider the right triangle with vertices at the focus, the center of the ellipse, and the tip of the semiminor axis. Its base is $a - r_{\mathrm{min}} = \epsilon a$ and its height is b. The hypotenuse is just a, since the total distance from one focus via the elliptical curve to the other focus is the same for any path. Then

$$\begin{aligned} a^2 &= \epsilon^2 a^2 + b^2 \\ b &= a\sqrt{1-\epsilon^2} \\ \mathcal{T} &= \frac{\pi a^2\sqrt{1-\epsilon^2}}{l/2\mu} \\ &= 2\pi a^2 \mu \sqrt{\frac{1-\epsilon^2}{l^2}} \\ &= \frac{2\pi a^2 \mu}{\sqrt{\mu k a}} \\ &= 2\pi a^{3/2}\sqrt{\frac{\mu}{k}}. \end{aligned} \tag{7.14}$$

(In the fifth equality, we used $l^2/(1-\epsilon^2) = \mu ka$ from Eq. (7.10).)

Equation (7.14), which is Kepler's third law, states that the orbital period is proportional to the $\frac{3}{2}$ power of the semimajor axis. Neglecting the tiny difference between the reduced mass μ and the actual planetary mass, to which k is proportional, this law predicts a planet's period given its orbit's semimajor axis together with the period and semimajor axis of any other planetary orbit. Its success was one of the experimental cornerstones of Newtonian mechanics.

7.7. Virial theorem.

For a circular orbit with a $1/r^2$ force, the familiar result $T = -E = -U/2$ is easily obtained from Eqs. (7.4) and (7.8) with $\dot{u} = 0$. For an elliptical orbit, both the potential and kinetic energies are functions of θ. To find a simple relation between T, E, and U, we must consider the time average values $\langle T \rangle$ and $\langle U \rangle$ of the kinetic and potential energies. For example, we may integrate

$$\begin{aligned} \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} U(t)\, dt &= \frac{1}{2\pi} \int_0^{2\pi} U(\theta) \frac{dt}{d\theta} d\theta \\ &= -\frac{1}{2\pi} \int_0^{2\pi} \frac{k}{r} \frac{\mu r^2}{l} d\theta. \end{aligned}$$

Inserting the elliptical orbit $r(\theta)$, the resulting integral is found in tables, yielding $\langle U \rangle = 2E$ as for the circular orbit.

A more elegant proof of this relation, with extension to a variety of power-law potentials, is provided by the *virial theorem*. Consider a system of N particles indexed by j. Define

$$\begin{aligned} S &\equiv \mathbf{p}_j \cdot \mathbf{r}_j \\ \frac{dS}{dt} &= \dot{\mathbf{p}}_j \cdot \mathbf{r}_j + \mathbf{p}_j \cdot \dot{\mathbf{r}}_j \\ &= \mathbf{F}_j \cdot \mathbf{r}_j + 2T, \end{aligned} \tag{7.15}$$

where $\mathbf{F}_j$ is the force on particle j, and T is the sum of the N kinetic energies.

Define the average value of dS/dt as

$$\langle \frac{dS}{dt} \rangle \equiv \lim_{t\to\infty} \frac{S(t) - S(0)}{t}.$$

This is zero if all the $\mathbf{p}_j$ and $\mathbf{r}_j$ are *bounded*, or if the motion is *periodic* with t chosen to be an integral multiple of the period. Under either condition, Eq. (7.15) requires

$$\langle T \rangle = -\tfrac{1}{2} \langle \mathbf{F}_j \cdot \mathbf{r}_j \rangle.$$

If $\mathbf{F}_j$ is derivable from a potential U_j which is proportional to r_j^{n+1},

$$\begin{aligned} \mathbf{F}_j = -\nabla U_j &= -\frac{(n+1) U_j \hat{r}_j}{|\mathbf{r}_j|} \\ \mathbf{F}_j \cdot \mathbf{r}_j &= -(n+1) U \\ \langle T \rangle &= \frac{n+1}{2} \langle U \rangle, \end{aligned} \tag{7.16}$$

where U is the sum of the N potential energies. With $n = -2$ for the gravitational or Coulomb potential,

$$\begin{aligned} \langle T \rangle &= -\tfrac{1}{2} \langle U \rangle \\ E &= \langle T + U \rangle = \langle T - 2T \rangle = -\langle T \rangle, \end{aligned}$$

as asserted above. Equation (7.16) is the *virial theorem*, widely used in classical kinetic theory.

7.8. Stability of circular orbits.

We define the *effective potential* $U_{\text{eff}}(r) \equiv U(r) + U'(r)$, where, as in section 7.3, $U'(r)$ is the pseudopotential $l^2/2\mu r^2$. Hamilton's equation in r becomes

$$\begin{aligned} \mathcal{H} &= \frac{p_r^2}{2\mu} + U_{\text{eff}}(r) \\ \dot{p}_r &= -\frac{\partial \mathcal{H}}{\partial r} = -\frac{dU_{\text{eff}}}{dr}. \end{aligned} \tag{7.17}$$

Therefore a circular orbit, for which $p_r \equiv 0$, is possible only at an extremum of U_{eff}. If $U_{\text{eff}}(r_0)$ is an extremum, close to that radius we may expand

$$\begin{aligned} \frac{dU_{\text{eff}}}{dr} &\approx (r - r_0) \frac{d^2 U_{\text{eff}}}{dr^2}\Big|_{r=r_0} \\ &\equiv k_{\text{eff}}(r - r_0) \\ \mu \ddot{r} = \dot{p}_r &= -k_{\text{eff}}(r - r_0). \end{aligned} \tag{7.18}$$

Near the extremum, if the second derivative k_{eff} is *positive* (U_{eff} is a *minimum*), $r - r_0$ experiences simple harmonic motion with angular frequency $\omega_\rho = (k_{\text{eff}}/\mu)^{1/2}$. Then the circular orbit is *stable*. Otherwise, $r - r_0$ grows exponentially, and the orbit is *unstable*.

As an alternative to the *method of effective potentials* described above, one may also analyze the stability of circular orbits using the *method of perturbations* as in section 3.7. As an example of both methods, we consider the motion of a particle of mass m in a gravitational field $\mathbf{g} = -g\hat{z}$. The particle is constrained to move on the frictionless surface of an upward-opening cone of half angle α.

In cylindrical coordinates (r, θ, z), the Lagrangian is

$$\mathcal{L} = T - U = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - mgz.$$

Applying the conical constraint $z = r\cot\alpha$, we may eliminate z:

$$\mathcal{L} = \tfrac{1}{2}m(\dot{r}^2\csc^2\alpha + r^2\dot{\theta}^2) - mgr\cot\alpha.$$

The coordinate θ is cyclic, so that its conjugate momentum $p_\theta = mr^2\dot{\theta} \equiv l$ is conserved. The Euler-Lagrange equation in r is

$$\begin{aligned} \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} &= \frac{\partial\mathcal{L}}{\partial r} \\ m\ddot{r}\csc^2\alpha &= mr\dot{\theta}^2 - mg\cot\alpha \\ \ddot{r} - \frac{l^2}{m^2r^3}\sin^2\alpha &= -g\sin\alpha\cos\alpha. \end{aligned} \tag{7.19}$$

Since $\partial\mathcal{L}/\partial t = 0$, and T is a quadratic function of $\dot{r}$ and $\dot{\theta}$,

$$\begin{aligned} E &= \mathcal{H} = T + U \\ &= \tfrac{1}{2}m(\dot{r}^2\csc^2\alpha + r^2\dot{\theta}^2) + mgr\cot\alpha \\ &= \tfrac{1}{2}m\dot{r}^2\csc^2\alpha + \frac{l^2}{2mr^2} + mgr\cot\alpha. \end{aligned} \tag{7.20}$$

The last two terms in the last equation sum to U_{eff}, which is infinite both at $r = 0$ and $r = \infty$. Therefore the intermediate extremum (at $r \equiv r_0$) must be a *minimum*, and a circular orbit at radius r_0 is *stable*.

Proceeding with the method of perturbations, in Eq. (7.19) we substitute $r = r_0 + \lambda\rho$, where λ is a small constant. Retaining only terms of 0^{th} order in λ, Eq. (7.19) becomes

$$-\frac{l^2}{m^2r_0^3}\sin^2\alpha = -g\sin\alpha\cos\alpha. \tag{7.21}$$

This may be solved for r_0. The remaining terms must also vanish:

$$\begin{aligned} 0 &= \lambda\ddot{\rho} - \frac{l^2\sin^2\alpha}{m^2r_0^3}\big((r_0/r)^3 - 1\big) \\ &= \lambda\ddot{\rho} - \frac{l^2\sin^2\alpha}{m^2r_0^3}\big((1 + \lambda\rho/r_0)^{-3} - 1\big) \\ &\approx \lambda\ddot{\rho} + \frac{l^2\sin^2\alpha}{m^2r_0^3}\frac{3\lambda\rho}{r_0} \\ &= \ddot{\rho} + \frac{3l^2\sin^2\alpha}{m^2r_0^4}\rho \\ &\equiv \ddot{\rho} + \omega_\rho^2\rho. \end{aligned} \tag{7.22}$$

In the third line we expanded $(1+\eta)^{-3} \approx 1 - 3\eta$ for $\eta \ll 1$, and in the last line we identified the angular frequency ω_ρ of radial oscillation.

The angular frequency of the unperturbed orbit is $\Omega = l/mr_0^2$. Therefore

$$\frac{\omega_\rho}{\Omega} = \sqrt{3}\sin\alpha.$$

The orbit is *closed* only for particular cone half angles such that ω_ρ/Ω is equal to a rational number m/n, where m and n are positive integers; n revolutions are required to close the orbit.

In this problem, the frequency of radial oscillation could have been obtained equally well using the method of effective potentials. But the method of perturbations is more general: a similar technique may be used to find the oscillation frequencies of deviations from unperturbed orbits which are *not* circular.

8. Collisions.

8.1. Elastic collisions.

In the absence of external forces, *all* collisions conserve total momentum (see section 2.4). However, not all collisions are *elastic*. Elastic collisions conserve total kinetic energy as well as momentum.

Within the plane of scattering, given the initial momenta, a two-body elastic collision may be characterized by a single quantity, for example the center of mass (C.M.) scattering angle. Since the transformation between nonrelativistic C.M. and laboratory angles is straightforward, specifying any single final state quantity in the laboratory (e.g. one final angle or energy) determines the whole problem. This can lead to a large set of assigned problems for which the algebra may be tedious but the conceptual basis remains elementary.

It is amusing that the elastic scattering of a nonrelativistic projectile from a stationary target of equal mass always results in an opening angle of $\pi/2$ between the two final state velocities. Label the projectile a, the stationary target b, and the final state particles c and d. According to the conservation laws,

$$\begin{aligned} &T_a + (T_b = 0) = T_c + T_d \\ &\qquad \Rightarrow \frac{1}{2m}(p_a^2 = p_c^2 + p_d^2) \\ &\mathbf{p}_a + (\mathbf{p}_b = 0) = \mathbf{p}_c + \mathbf{p}_d \\ &\qquad\qquad \Rightarrow p_a^2 = p_c^2 + p_d^2 + 2\mathbf{p}_c \cdot \mathbf{p}_d. \end{aligned}$$

For the second and fourth equalities to be consistent, $\mathbf{p}_c$ and $\mathbf{p}_d$ must be mutually perpendicular.

8.2. Rutherford scattering.

By themselves, momentum and energy conservation determine the result of an elastic scattering only if one final state quantity also can be supplied. However, if the force law is known, the final state can be predicted from the initial conditions alone. The classic example is *Rutherford scattering* of two charged particles under the influence of the Coulomb force. In the C.M., consider a nonrelativistic projectile of charge ze impinging with initial relative velocity v_0 upon a target of charge Ze. The electrostatic force between them is centrally directed and of size Zze^2/r^2 in Gaussian units.

This is not enough information to predict the scattered final state, as we have not yet specified whether the collision is "grazing" or "head-on". The missing quantity is the *impact parameter* $b \equiv l/\mu v_0$. It is named for the fact that the undeflected path of the projectile misses the target by the distance b. In terms of these quantities, the constants introduced in section 7.5 become

$$k = -Zze^2 \quad E = \tfrac{1}{2}\mu v_0^2 \quad l = \mu b v_0.$$

Again defining $\theta \equiv \theta_0$ at the perigee $r_{\min}$, Eq. (7.9) requires $\epsilon < -1$ when k is negative, as it is here. Solving the first line of Eq. (7.12) for $r_{\min}$,

$$\begin{aligned} 0 &= \frac{l^2}{2\mu} r_{\min}^{-2} - k r_{\min}^{-1} - E \\ r_{\min}^{-1} &= \frac{k \pm \sqrt{k^2 + 2El^2/\mu}}{l^2/\mu} \\ &= \frac{-k\mu}{l^2}(\sqrt{1+\eta^2} - 1), \end{aligned} \tag{8.1}$$

where

$$\eta \equiv \sqrt{\frac{2El^2}{\mu k^2}} = \frac{\mu b v_0^2}{Zze^2}. \tag{8.2}$$

From Eq. (7.10),

$$r_{\min}^{-1} = \frac{-k\mu}{l^2}(-\epsilon - 1). \tag{8.3}$$

Comparing Eqs. (8.1) and (8.3),

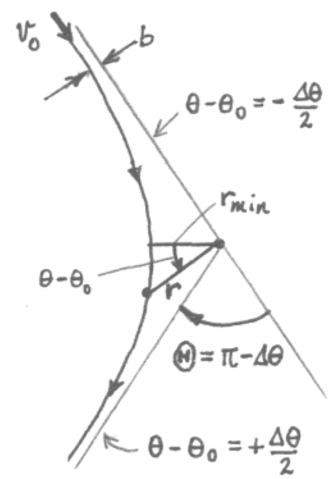
$$-\epsilon = \sqrt{1+\eta^2}. \tag{8.4}$$

To keep r in Eq. (7.9) positive while $k < 0$, we require

$$\begin{aligned} -\epsilon^{-1} &\le \cos(\theta - \theta_0) \\ \cos^{-1}(1+\eta^2)^{-1/2} &\le |\theta - \theta_0| \\ \tan^{-1}\eta &\le |\theta - \theta_0|. \end{aligned} \tag{8.5}$$

Thus $\theta - \theta_0$ varies only within a range

$$\Delta\theta = 2\tan^{-1}\eta. \tag{8.6}$$



The scattering angle Θ is defined as $\Theta \equiv \pi - \Delta\theta$, so that $\Theta = 0$ for a grazing collision and $\Theta = \pi$ for a head-on collision. Then, combining Eqs. (8.6) and (8.2),

$$\begin{aligned} \Theta/2 &= \pi/2 - \Delta\theta/2 \\ &= \pi/2 - \tan^{-1}\eta \\ &= \cot^{-1}\eta \\ &= \tan^{-1}\frac{Zze^2}{\mu b v_0^2} \\ b &= \frac{Zze^2}{\mu v_0^2}\cot\Theta/2. \end{aligned} \tag{8.7}$$

This last equation connects the impact parameter to the scattering angle.

8.3. Scattering cross section.

In Rutherford's experiment, the projectiles were ^{4}He nuclei and the targets were ^{197}Au nuclei in a thin foil. The detectors were dark-adapted students barely observing flashes in scintillating plates. Like more modern instruments, the students could measure the scattering angle Θ, but not the tiny impact parameter b. To interpret his data, Rutherford needed to predict the *distribution* of scattering angles.

Let Γ be the flux of incident projectiles (particles/cm^2-sec), and let dN be the number of incident projectiles/sec with impact parameter between b and $b + db$ and with azimuth between ϕ and $\phi + d\phi$. Using Eq. (8.7),

$$\begin{aligned} dN &= \Gamma b\, db\, d\phi \\ \frac{dN}{\Gamma} &= \Big(\frac{Zze^2}{\mu v_0^2}\Big)^2 \cot\tfrac{\Theta}{2}\, d\cot\tfrac{\Theta}{2}\, d\phi \\ &= \Big(\frac{Zze^2}{\mu v_0^2}\Big)^2 \cot\tfrac{\Theta}{2}\csc^2\tfrac{\Theta}{2}\, d\tfrac{\Theta}{2}\, d\phi \\ &= \Big(\frac{Zze^2}{\mu v_0^2}\Big)^2 \cos\tfrac{\Theta}{2}\sin\tfrac{\Theta}{2}\csc^4\tfrac{\Theta}{2}\, d\tfrac{\Theta}{2}\, d\phi \\ &= \Big(\frac{Zze^2}{2\mu v_0^2}\Big)^2 \csc^4\tfrac{\Theta}{2}\, d\Omega, \end{aligned}$$

where $d\Omega \equiv \sin\Theta\, d\Theta\, d\phi = 4\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}\, d\frac{\Theta}{2}\, d\phi$ is an infinitesimal element of *solid angle*. The quantity $dN/\Gamma d\Omega$ has the dimensions of an *area* and is called the *differential scattering cross section $d\sigma/d\Omega$*:

$$\frac{d\sigma}{d\Omega} = \Big(\frac{Zze^2}{2\mu v_0^2}\Big)^2 \frac{1}{\sin^4\Theta/2}. \tag{8.8}$$

This is the famous differential cross section for Rutherford scattering. Physically, $d\sigma$ is the cross sectional area of the beam which is scattered by the target through angle Θ into an element $d\Omega$ of solid angle. The probability for a single ^{4}He nucleus to scatter into $d\Omega$ is $\Sigma(d\sigma/d\Omega)d\Omega$, where Σ is the number of ^{197}Au nuclei per *square* cm of target foil.

Rutherford scattering has special properties that deserve comment. First, like the gravitational force, the $1/r^2$ Coulomb force is said to have *infinite range* because the *total* scattering cross section σ_T is infinite:

$$\sigma_T \equiv 2\pi\int_0^\pi \frac{d\sigma}{d\Omega}\sin\Theta d\Theta \propto \int_0^\pi \frac{\sin\Theta d\Theta}{\sin^4\Theta/2} = \infty.$$

Second, in the quantum mechanical sense, the individual charges on the projectile and target nuclei act *coherently* to produce the scattering. This follows from the fact that the differential cross section is proportional to the *square* of z and of Z. (In fact, the quantum mechanical calculation for Rutherford scattering yields the identical result.) The cross section is independent of the sign of either charge.

The final comment is that $d\sigma/d\Omega$ is *finite* at $\Theta = \pi$: per unit solid angle, there is a finite probability for *backscattering*. Using this striking prediction, Rutherford was able experimentally to distinguish between a nucleus modeled to be small in extent, relative to $r_{\min}$, and another more extended model for which backscattering could not occur. At ever higher projectile energies, scattering of simple projectiles by ever smaller targets has provided fundamental insights. In the 1950's, elastic scattering of $\approx 10^8$ eV electrons determined the size and shape of nuclei in exquisite detail. In 1968, quarks were discovered in the inelastic scattering of $\approx 10^{10}$ eV electrons from protons.

9. Rotational motion.

9.1. Pseudoforces in rotating systems.

The present discussion builds on the results of section 1. Recall that $\tilde{x}$ is a column vector representing a vector in the body (rotating) system, while $\tilde{x}'$ is the same vector in an inertial (fixed) system. At $t = 0$ the two coordinate systems are coincident, so that $\tilde{x}(0) = \tilde{x}'(0)$. At other times, $\tilde{x}$ and $\tilde{x}'$ are related by a rotation, which is represented by an orthogonal 3×3 matrix Λ:

$$\tilde{x}' = \Lambda \tilde{x}. \tag{9.1}$$

If the body system is rotating relative to the inertial system with angular velocity $\vec{\omega}$, and $\mathbf{P}$ ($\mathbf{P}'$) is the same vector observed in the rotating (fixed) system, recall that $d\mathbf{P}'/dt$ acquires an extra term due to the rotation:

$$\frac{d\mathbf{P}'}{dt} = \frac{d\mathbf{P}}{dt} + \vec{\omega} \times \mathbf{P}. \tag{9.2}$$

Choosing $\mathbf{P}^{(\prime)} = \mathbf{r}^{(\prime)}$, Eq. (9.2) becomes

$$\mathbf{v}' = \mathbf{v} + \vec{\omega} \times \mathbf{r}. \tag{9.3}$$

Alternatively, choosing $\mathbf{P}'$ to be the left hand side of Eq. (9.3), and $\mathbf{P}$ to be the right hand side, Eq. (9.2) becomes

$$\begin{aligned} \frac{d\mathbf{v}'}{dt} &= \Big(\frac{d}{dt} + \vec{\omega}\times\Big)\big(\mathbf{v} + \vec{\omega} \times \mathbf{r}\big) \\ \frac{\mathbf{F}'}{m} &= \frac{d\mathbf{v}}{dt} + 2\vec{\omega} \times \mathbf{v} + \vec{\omega} \times (\vec{\omega} \times \mathbf{r}) \\ m\frac{d\mathbf{v}}{dt} &= \mathbf{F}' - 2m\vec{\omega} \times \mathbf{v} - m\vec{\omega} \times (\vec{\omega} \times \mathbf{r}) \\ &\equiv \mathbf{F}' + \mathbf{F}_{\text{Coriolis}} + \mathbf{F}_{\text{centrifugal}}. \end{aligned} \tag{9.4}$$

When observed in the body frame, the mass accelerates as though under the influence of *pseudoforces* $\mathbf{F}_{\text{Coriolis}}$ and $\mathbf{F}_{\text{centrifugal}}$, in addition to the *actual* force $\mathbf{F}'$ acting in the inertial frame.

On the surface of the earth, which spins with angular velocity $\vec{\omega}_e$ directed out of the north pole, the centrifugal force points outward from the axis with magnitude $m\omega_e^2 r_e \sin\lambda = 0.003455 mg \sin\lambda$, where r_e is the earth's radius, λ is the colatitude (measured from the north pole) and g is the gravitational acceleration at the surface. The magnitude of the effective (apparent) acceleration $\mathbf{g}_{\text{eff}} \equiv \mathbf{g} + \mathbf{F}_{\text{centrifugal}}/m$ is slightly reduced, especially for colatitudes near $\pi/2$. This causes the earth to bulge near the equator. The direction of $\mathbf{g}_{\text{eff}}$ also shifts slightly. For example, in the northern hemisphere, $\mathbf{g}_{\text{eff}}$ intersects the earth's axis slightly south of its center.

The Coriolis force is proportional to velocity. In the northern hemisphere ($0 < \lambda < \pi/2$), falling bodies experience a force $2m\omega_e v \sin\lambda$ to the *east*, and bodies moving horizontally feel a Coriolis force whose *horizontal* component is $2m\omega_e v \cos\lambda$ to the *right*. In the southern hemisphere, $\cos\lambda$ reverses sign and the horizontal Coriolis force is to the *left*. The Coriolis force accounts *e.g.* for the counterclockwise circulation of storms in the northern hemisphere: as an element of air is sucked into the low-pressure eye, it veers right. As it happened, gunnery tables available to the U.S. Navy at the time of early south Pacific battles in WW II were written only for the northern hemisphere. The seriousness of this blunder was diminished by the emerging dominance of air power.

9.2. Foucault pendulum.

If a freely pivoting pendulum is placed at the north pole, as viewed in an inertial frame it oscillates in a fixed plane while the earth rotates underneath it. Correspondingly, on the earth, its plane of oscillation is observed to precess with angular velocity $-\vec{\omega}_e$. If the pendulum is placed at the equator, its plane is not observed to precess with respect to the earth's surface, because the equator is a symmetry point at which a preferred direction for this precession cannot be identified. More generally, if this *Foucault pendulum* is placed at colatitude λ, its plane of oscillation precesses clockwise ("CW") relative to the earth's surface with angular velocity $\omega_e \cos\lambda$. At least, this guess is consistent with the two limiting cases just considered.

To analyze this problem further, consider a local (unprimed) coordinate system with its origin on the earth's surface at colatitude λ. We take the $\hat{z}$ direction to be normal to the surface (ignoring centrifugal forces). In this frame, ω_e lies in the z-north plane at angle λ to $\hat{z}$. Including the Coriolis force, the pendulum bob satisfies the equation

$$\begin{aligned}\ddot{\mathbf{r}} &= -g\hat{z} + \frac{\mathbf{T}}{m} - 2\vec{\omega}_e \times \dot{\mathbf{r}} \\ &\approx -g\hat{z} + \frac{\mathbf{T}}{m} - 2\omega_e \cos\lambda\, \hat{z} \times \dot{\mathbf{r}},\end{aligned}$$

where $\mathbf{T}$ is the string tension. In the second equality we used the fact that $\dot{\mathbf{r}}$ lies mainly in the horizontal plane, assuming small-angle oscillations.

The final step is to analyze the same problem in a (starred) system having the same origin, which is precessing CW *with respect to the earth's surface* with angular velocity $\vec{\Omega} = -\hat{z}\omega_e \cos\lambda$. The equation of motion of the pendulum bob becomes

$$\ddot{\mathbf{r}}^* = -g\hat{z} + \frac{\mathbf{T}}{m} - 2\omega_e \cos\lambda\, \hat{z} \times \dot{\mathbf{r}}^* - 2\vec{\Omega} \times \dot{\mathbf{r}}^*.$$

The last two terms cancel, and the resulting equation is that of a simple pendulum. In the starred system, the plane of oscillation does not precess. Therefore, in the *unprimed* system, the plane of oscillation precesses CW with the same angular velocity $\omega_e \cos\lambda$ as does the starred system. This justifies the original assertion.

Large Foucault pendula with slowly precessing planes of oscillation are staples of many science museums. By observing the angle through which the pendulum has precessed, you can monitor the duration of your visit, or, with the help of a watch, measure the museum's latitude.

9.3. Angular velocity from Euler rotations.

Recall from section 1 that the Euler rotation Λ^t is a transformation from the space (inertial) frame to the body frame, by means of three successive transformations:

$$\begin{aligned}\tilde{x} &= \Lambda^t_\psi \tilde{x}''' \\ &= \Lambda^t_\psi \Lambda^t_\theta \tilde{x}'' \\ &= \Lambda^t_\psi \Lambda^t_\theta \Lambda^t_\phi \tilde{x}' \\ &\equiv \Lambda^t \tilde{x}',\end{aligned} \tag{9.5}$$

where Λ^t_ψ, Λ^t_θ, and Λ^t_ϕ are defined in Eq. (1.9). In order to solve problems in the body system, we need to compute the Cartesian components there of the angular velocity $\vec{\omega}$ that results when the Euler angles ϕ, θ, and ψ vary with time.

The angular velocities in question are directed along the $3'$, $1''$, and $3'''$ axes, respectively. Transforming all three to the body system,

$$\begin{aligned}\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} &= \Lambda^t_\psi \Lambda^t_\theta \Lambda^t_\phi \begin{pmatrix}0\\0\\ \dot{\phi}\end{pmatrix} + \\ &\quad + \Lambda^t_\psi \Lambda^t_\theta \begin{pmatrix}\dot{\theta}\\0\\0\end{pmatrix} + \Lambda^t_\psi \begin{pmatrix}0\\0\\ \dot{\psi}\end{pmatrix} \\ &= \begin{pmatrix}\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi}\end{pmatrix}.\end{aligned} \tag{9.6}$$

We shall use this relation in the study of tops.

9.4. Angular momentum of a rigid body.

About the origin, N point masses at *space* coordinates $\mathbf{r}'_i$ $(1 \le i \le N)$ have combined angular momentum

$$\mathbf{L} = \sum_{i=1}^{N} m_i(\mathbf{r}'_i \times \dot{\mathbf{r}}'_i). \tag{9.7}$$

We would like to express $\mathbf{L}$ in the *body* system of coordinates. In that system, if the point masses make up a *rigid body*, we can take advantage of whatever symmetries the body might possess.

If the motion of the body system is limited to a simple *rotation* characterized by angular velocity $\vec{\omega}$, we choose the origins of the space and body axes to be the same. In addition, we choose to consider $\mathbf{L}$ at an instant of time when the two systems are entirely coincident. This means that $\mathbf{r}_i = \mathbf{r}'_i$ for all i at that time, but of course $\dot{\mathbf{r}}_i \ne \dot{\mathbf{r}}'_i$. Assuming that the body is rigid, $\mathbf{r}_i$ is a constant in the body system. Equation (9.3) becomes

$$\dot{\mathbf{r}}'_i = \vec{\omega} \times \mathbf{r}_i. \tag{9.8}$$

The angular momentum may be written

$$\begin{aligned}\mathbf{L} &= \sum_i m_i\big(\mathbf{r}_i \times (\vec{\omega} \times \mathbf{r}_i)\big) \\ &= \sum_i m_i\big(\vec{\omega}(\mathbf{r}_i \cdot \mathbf{r}_i) - \mathbf{r}_i(\mathbf{r}_i \cdot \vec{\omega})\big).\end{aligned} \tag{9.9}$$

Changing to matrix notation in Cartesian coordinates, with $\tilde{L}$, $\tilde{\mathbf{r}}$, and $\tilde{\omega}$ representing column vectors, and with I as the unit matrix,

$$\begin{aligned}\tilde{L} &= \sum_i m_i\big(r_i^2\tilde{\omega} - \tilde{r}_i(\tilde{r}_i^t\tilde{\omega})\big) \\ &= \sum_i m_i\big(r_i^2\mathrm{I}\tilde{\omega} - (\tilde{r}_i\tilde{r}_i^t)\tilde{\omega}\big) \\ &= \big(\sum_i m_i(r_i^2\mathrm{I} - \tilde{r}_i\tilde{r}_i^t)\big)\tilde{\omega} \\ &\equiv \mathcal{I}\tilde{\omega} \\ \mathcal{I} &\equiv \sum_i m_i\big(r_i^2\mathrm{I} - \tilde{r}_i\tilde{r}_i^t\big).\end{aligned} \tag{9.10}$$

In the above we have introduced the *inertia tensor* $\mathcal{I}$, which relates the angular momentum to the angular velocity of a rigid body.

Because the matrix notation in Eq. (9.10) can be cryptic, it is useful to display $\mathcal{I}$ in component form:

$$\begin{aligned}\tilde{r}_i\tilde{r}_i^t &= \begin{pmatrix} x_1^ix_1^i & x_1^ix_2^i & x_1^ix_3^i \\ x_2^ix_1^i & x_2^ix_2^i & x_2^ix_3^i \\ x_3^ix_1^i & x_3^ix_2^i & x_3^ix_3^i \end{pmatrix} \\ \mathcal{I}_{jk} &\equiv \sum_i m_i\big(\delta_{jk}r_i^2 - x_j^ix_k^i\big),\end{aligned} \tag{9.11}$$

where e.g. x_j^i is the j^{th} component $(1 \le j \le 3)$ of the coordinate $\mathbf{r}_i$ of the i^{th} mass. For example,

$$\begin{aligned}\mathcal{I}_{11} &= \sum_i m_i\big(r_i^2 - (x_1^i)^2\big) \\ &= \sum_i m_i\big((x_2^i)^2 + (x_3^i)^2\big) \\ \mathcal{I}_{12} &= -\sum_i m_ix_1^ix_2^i.\end{aligned}$$

What makes $\mathcal{I}$ a *tensor* is its transformation property. If $\tilde{x}$ is a vector transforming from the body to the space system according to $\tilde{x}' = \Lambda\tilde{x}$, the inertia tensor transforms according to the *similarity transformation:*

$$\mathcal{I}' = \Lambda\mathcal{I}\Lambda^t. \tag{9.12}$$

9.5. Elementary properties of the inertia tensor.

Obviously, since $\mathcal{I}$ rarely is proportional to the unit matrix, for arbitrary $\hat{\omega}$ the angular momentum $\mathbf{L} = \mathcal{I}\vec{\omega}$ rarely is directed along $\hat{\omega}$. It is true that $\mathbf{L} \parallel \vec{\omega}$ in many elementary problems; this occurs when $\vec{\omega}$ is directed along one of the *principal axes* of the body, defined later on.

A simple example with $\mathbf{L}$ not parallel to $\vec{\omega}$ is a barbell rotating about a vertical axis through its center. The bar is inclined at an angle ψ to the horizontal, so that one weight orbits the axis in an higher plane than does the other. For each weight, $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ must be $\perp$ to the bar, so $\mathbf{L}$ must make the same angle ψ with the axis of rotation.

When $\mathbf{L} = I\vec{\omega}$, where I is the scalar moment of inertia, the kinetic energy of rotation is $T_{\text{rot}} = \frac{1}{2}I\omega^2$. More generally,

$$\begin{aligned}T_{\text{rot}} &= \tfrac{1}{2}\sum_i m_i(v_i')^2 \\ &= \tfrac{1}{2}\sum_i m_i\mathbf{v}_i' \cdot \frac{d\mathbf{r}_i'}{dt} \\ &= \tfrac{1}{2}\sum_i m_i\mathbf{v}_i' \cdot (\vec{\omega} \times \mathbf{r}_i') \\ &= \tfrac{1}{2}\sum_i m_i\vec{\omega} \cdot (\mathbf{r}_i' \times \mathbf{v}_i') \\ &= \tfrac{1}{2}\vec{\omega} \cdot \big(\sum_i m_i\mathbf{r}_i' \times \mathbf{v}_i'\big) \\ &= \tfrac{1}{2}\vec{\omega} \cdot \mathbf{L} \\ &= \tfrac{1}{2}\vec{\omega} \cdot (\mathcal{I}\vec{\omega}) \\ &= \tfrac{1}{2}\tilde{\omega}^t\mathcal{I}\tilde{\omega}.\end{aligned} \tag{9.12}$$

Defining $\tilde{n}$ to be a unit column vector along $\vec{\omega}$, Eq. (9.12) becomes

$$\begin{aligned}T_{\text{rot}} &= \tfrac{1}{2}\omega^2\tilde{n}^t\mathcal{I}\tilde{n} \\ &\equiv \tfrac{1}{2}I\omega^2 \\ I &\equiv \tilde{n}^t\mathcal{I}\tilde{n}.\end{aligned} \tag{9.13}$$

As a bonus, Eq. (9.13) relates the inertia tensor $\mathcal{I}$ to the scalar moment of inertia I about a particular axis of rotation. However, while useful for calculating $T_{\rm rot}$, this scalar I obviously *cannot* be employed to relate $\mathbf{L}$ to $\vec{\omega}$ in the general case.

So far we have been considering a rigid body to be a collection of discrete masses, for which the inertia tensor is given by Eq. (9.11), repeated below. The same equation for a *continuous* mass distribution is obtained by replacing the sum over m_i by an integral over the mass density ρ:

$$\begin{aligned}\mathcal{I}_{jk} &= \sum_i m_i\big(\delta_{jk}\sum_{l=1}^{3}(x_l^i)^2 - x_j^i x_k^i\big)\\ &= \int dx_1\int dx_2\int dx_3\Big(\rho(x_1,x_2,x_3)\times\\ &\times\big(\delta_{jk}\sum_{l=1}^{3}(x_l)^2 - x_j x_k\big)\Big).\end{aligned} \tag{9.14}$$

The integral is taken over the entire volume of the rigid body. It is usually easy to evaluate unless the boundaries of the body are not readily expressible in terms of the Cartesian coordinates x_1, x_2, and x_3. In that event, it may be necessary to transform the integral to cylindrical or spherical coordinates.

9.6. Diagonalization of the inertia tensor.

In this section we prove, for *any* rigid body, that there exist three orthogonal unit vectors $\tilde{s}_r$, $1 \le r \le 3$, such that

$$\mathcal{I}\tilde{s}_r = e_r\tilde{s}_r, \tag{9.15}$$

where e_r is a real constant called the $r^{\rm th}$ *eigenvalue*. The $\tilde{s}_r$ are the *eigenvectors* – the directions of the body's *principal axes*. When the rigid body rotates about any of its principal axes, $\mathbf{L}$ and $\vec{\omega}$ are *parallel*:

$$\tilde{L} = \mathcal{I}\tilde{\omega} = \mathcal{I}(\omega\tilde{s}_r) = e_r(\omega\tilde{s}_r) = e_r\tilde{\omega}.$$

(Summation over r is *never* implied!) This simple relationship between $\mathbf{L}$ and $\vec{\omega}$ is so advantageous that we almost always use the principal axis frame when we work in the body system.

If the eigenvalue equation (9.15) is satisfied,

$$\begin{aligned}0 &= \mathcal{I}\tilde{s} - e\tilde{s} = (\mathcal{I} - e\mathrm{I})\tilde{s}\\ &= (\mathcal{I} - e\mathrm{I})\tilde{s}s\\ &= \det\big((\mathcal{I} - e\mathrm{I})\tilde{s}s\big)\\ &= \det(\mathcal{I} - e\mathrm{I})\det(\tilde{s}s)\\ 0 &= \det(\mathcal{I} - e\mathrm{I}).\end{aligned} \tag{9.16}$$

This *"secular equation"* is a cubic equation with three roots e_1, e_2, and e_3. Once it is solved to yield these eigenvalues e_r, the eigenvectors $\tilde{s}_r$ are obtained from the equations

$$\begin{aligned}(\mathcal{I} - e_r\mathrm{I})\tilde{s}_r &= 0\\ \tilde{s}_r^t\tilde{s}_r &= 1.\end{aligned} \tag{9.17}$$

The latter equality ensures that the eigenvectors are of unit length.

First we prove that the eigenvalues of $\mathcal{I}$ are *real*. Here Eq. (9.11) reminds us that $\mathcal{I}$ is *real* and *symmetric*, and therefore *self-adjoint*: $\mathcal{I}^\dagger = \mathcal{I}$. For brevity dropping the index r,

$$\begin{aligned}\mathcal{I}\tilde{s} &= e\tilde{s}\\ \tilde{s}^\dagger\mathcal{I}\tilde{s} &= e\tilde{s}^\dagger\tilde{s}\\ (\tilde{s}^\dagger\mathcal{I}\tilde{s})^\dagger &= (e\tilde{s}^\dagger\tilde{s})^\dagger\\ \tilde{s}^\dagger(\tilde{s}^\dagger\mathcal{I})^\dagger &= e^*\tilde{s}^\dagger\tilde{s}\\ \tilde{s}^\dagger\mathcal{I}^\dagger\tilde{s} &= e^*\tilde{s}^\dagger\tilde{s}\\ \tilde{s}^\dagger\mathcal{I}\tilde{s} &= e^*\tilde{s}^\dagger\tilde{s}\\ e\tilde{s}^\dagger\tilde{s} &= e^*\tilde{s}^\dagger\tilde{s}\\ e &= e^*.\end{aligned}$$

Obviously the eigenvectors $\tilde{s}_r$ also can be defined to be real, since everything else in the linear Eq. (9.15) is real.

Next we prove that two eigenvectors $\tilde{s}_1$ and $\tilde{s}_2$ corresponding to two different eigenvalues e_1

and e_2 are *orthogonal*:

$$\begin{aligned}
e_1 &\neq e_2 \\
\mathcal{I}\tilde{s}_1 &= e_1\tilde{s}_1 \\
\mathcal{I}\tilde{s}_2 &= e_2\tilde{s}_2 \\
\tilde{s}_2^t\mathcal{I}\tilde{s}_1 &= e_1\tilde{s}_2^t\tilde{s}_1 \\
\tilde{s}_1^t\mathcal{I}\tilde{s}_2 &= e_2\tilde{s}_1^t\tilde{s}_2 \\
(\tilde{s}_1^t\mathcal{I}\tilde{s}_2)^t &= (e_2\tilde{s}_1^t\tilde{s}_2)^t \\
\tilde{s}_2^t\mathcal{I}^t\tilde{s}_1 &= e_2\tilde{s}_2^t\tilde{s}_1 \\
\tilde{s}_2^t\mathcal{I}\tilde{s}_1 &= e_2\tilde{s}_2^t\tilde{s}_1 \\
0 &= (e_2 - e_1)\tilde{s}_2^t\tilde{s}_1 \\
0 &= \tilde{s}_2^t\tilde{s}_1 .
\end{aligned}$$

If the same eigenvalue is shared by two or more different eigenvectors, it is easy to form linear combinations of them to construct three mutually orthogonal eigenvectors. Thus *the principal axes of a rigid body are always orthogonal*.

9.7. Principal axis transformation.

For simplicity, arrange the eigenvector signs so that $\mathbf{s}_1 \times \mathbf{s}_2 = \mathbf{s}_3$. The reference frame defined by these three unit vectors is called the *principal axis frame*. Denote it by two primes ($''$). A vector $\tilde{x}''$ in the principal axis frame is related to the same vector $\tilde{x}$ in the (unprimed) body frame, in which $\mathcal{I}$ was originally calculated, by a *principal axis transformation*:

$$\tilde{x}'' = \Lambda\tilde{x} .$$

Since $\mathcal{I}$ is a tensor, $\mathcal{I}''$ is related to $\mathcal{I}$ by a similarity transformation:

$$\begin{aligned}
\mathcal{I}'' &= \Lambda\mathcal{I}\Lambda^t \\
\Lambda^t\mathcal{I}'' &= \mathcal{I}\Lambda^t
\end{aligned}$$

$$\begin{pmatrix} \Lambda_{11}^t & \Lambda_{12}^t & \Lambda_{13}^t \\ \Lambda_{21}^t & \Lambda_{22}^t & \Lambda_{23}^t \\ \Lambda_{31}^t & \Lambda_{32}^t & \Lambda_{33}^t \end{pmatrix} \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix} =$$

$$= \mathcal{I} \begin{pmatrix} \Lambda_{11}^t & \Lambda_{12}^t & \Lambda_{13}^t \\ \Lambda_{21}^t & \Lambda_{22}^t & \Lambda_{23}^t \\ \Lambda_{31}^t & \Lambda_{32}^t & \Lambda_{33}^t \end{pmatrix} . \tag{9.18}$$

The first column of Eq. (9.19) is just the first eigenvalue equation:

$$e_1 \begin{pmatrix} \Lambda_{11}^t \\ \Lambda_{21}^t \\ \Lambda_{31}^t \end{pmatrix} = \mathcal{I} \begin{pmatrix} \Lambda_{11}^t \\ \Lambda_{21}^t \\ \Lambda_{31}^t \end{pmatrix} . \tag{9.19}$$

Comparing Eq. (9.19) to Eq. (9.15), we see that the column vector on either side of Eq. (9.19) is proportional to the first eigenvector $\tilde{s}_1$. Since the transformation is orthogonal ($\Lambda^t\Lambda = \mathrm{I}$), this constant of proportionality is unity. Therefore the transpose Λ^t of the principal axis transformation matrix is the *matrix which has columns equal to the eigenvectors*. That is, Λ_{ir}^t is the i^{th} component of the r^{th} eigenvector $\tilde{s}_r$.

To recapitulate, the inertia tensor is diagonalized by an orthogonal transformation $\mathcal{I}'' = \Lambda\mathcal{I}\Lambda^t$, where Λ^t is the matrix whose columns are the eigenvectors. The diagonals of the transformed inertia tensor are the eigenvalues.

One may solve inertia tensor problems using various strategies. In one method, calculate $\mathcal{I}$ in a convenient coordinate system, solve the secular equation (9.16) for the eigenvalues, then solve (9.17) for the eigenvectors. This yields the matrix Λ^t, which facilitates a transformation to the principal axis system.

Alternatively, the principal axes of the rigid body may be obvious from symmetry considerations. For example, Eq. (9.11) guarantees that the normal to a plane of symmetry is a principal axis. So is an axis of cylindrical symmetry. Often you can guess the principal axes and calculate the inertia tensor in the principal axis system directly.

9.8. Parallel axis theorem.

Usually it is desirable to know the inertia tensor $\mathcal{I}_{\text{CM}}$ in a system of (body) coordinates with its origin at the C.M. In this system, there is no translational motion of the C.M.; the kinetic energy arises simply from rotation, and is equal to $T = \tilde{\omega}^t\mathcal{I}\tilde{\omega}$, as in Eq. (9.12).

On the other hand, it is often far more convenient to calculate $\mathcal{I}$ in a new (body) system in which the axes have the same orientation, but the origin is displaced from the C.M. Denote by $\mathbf{R}$ the vector *from* the new origin *to* the C.M.

Using $\mathbf{r}_i$ as the coordinate of a mass point in the new system, and $\mathbf{r}_i^*$ as the C.M. coordinate of the same point,

$$\begin{aligned}
\mathcal{I} &= \sum_i m_i(r_i^2 \mathrm{I} - \tilde{r}_i \tilde{r}_i^t) \\
&= \sum_i m_i\big((\mathbf{r}_i^* + \mathbf{R})^2 \mathrm{I} - (\tilde{r}_i^* + \tilde{R})(\tilde{r}_i^{*t} + \tilde{R}^t)\big) \\
&= \sum_i m_i\big((r_i^{*2} + R^2)\mathrm{I} - (\tilde{r}_i^* \tilde{r}_i^{*t} + \tilde{R}\tilde{R}^t)\big) \\
&= \mathcal{I}^{CM} + M(R^2 \mathrm{I} - \tilde{R}\tilde{R}^t) \text{ , or} \\
\mathcal{I}_{jk} &= \mathcal{I}_{jk}^{CM} + M(R^2 \delta_{jk} - R_j R_k) \, ,
\end{aligned} \tag{9.20}$$

where M is the total mass. The cross terms in the second equality vanished in the third because $\sum_i m_i \mathbf{r}_i^* \equiv 0$. The last two equations say that the inertia tensor at $\mathbf{R}$ is the inertia tensor at the center of mass plus the inertia tensor of a point mass M located at $\mathbf{R}$.

Equation (9.20) is the *parallel axis theorem*. With it one may easily calculate $\mathcal{I}_{CM}$ given $\mathcal{I}$, or *vice versa*.

According to Eq. (9.13), the scalar moment of inertia I about an axis $\hat{n}$ is

$$\begin{aligned}
I &= \tilde{n}^t \mathcal{I} \tilde{n} \\
&= \tilde{n}^t \mathcal{I}_{CM} \tilde{n} + \tilde{n}^t M(R^2 \mathrm{I} - \tilde{R}\tilde{R}^t)\tilde{n} \\
&= I_{CM} + M(R^2 - \tilde{n}^t \tilde{R}\tilde{R}^t \tilde{n}) \\
&= I_{CM} + M(R^2 - \hat{n}\cdot\mathbf{R}\,\mathbf{R}\cdot\hat{n}) \\
&= I_{CM} + M(R^2 - R_n^2) \\
&= I_{CM} + M(\mathbf{R} \times \hat{n})^2.
\end{aligned} \tag{9.21}$$

Here we have introduced I_{CM}, the scalar moment of inertia for rotation about a parallel axis through the C.M. If $\hat{n}$ is *not* a principal axis, recall that such scalar moments of inertia are useful only for calculating the rotational kinetic energy. If, on the other hand, $\hat{n}$ *is* a principal axis, I and I_{CM} *do* relate $\mathbf{L}$ to $\vec{\omega}$, and Eq. (9.21) is equivalent to the simple form of the parallel axis theorem that is usually found in introductory courses.

10. Euler's equations for rotational motion.

10.1. Evolution of the angular velocity.

Consider the rotation of a free rigid body ($U = 0$) about its C.M., using the body's Euler angles ϕ, θ, and ψ as generalized coordinates. The Lagrangian reduces to

$$\mathcal{L} = T = \tfrac{1}{2}(\omega_1^2 \mathcal{I}_{11} + \omega_2^2 \mathcal{I}_{22} + \omega_3^2 \mathcal{I}_{33})$$

if $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are chosen to be the principal axes. Using Eq. (9.6), it would be possible, though messy, to express T in terms of the Euler angles and their time derivatives. Alternatively, to obtain one Euler-Lagrange equation, it is easier to write

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} &= \frac{\partial \mathcal{L}}{\partial \psi} \\
\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \omega_3}\frac{\partial \omega_3}{\partial \dot{\psi}}\Big) &= \frac{\partial \mathcal{L}}{\partial \omega_1}\frac{\partial \omega_1}{\partial \psi} + \frac{\partial \mathcal{L}}{\partial \omega_2}\frac{\partial \omega_2}{\partial \psi}.
\end{aligned}$$

In the second equality we have used the fact that only ω_3 depends on $\dot{\psi}$, and only ω_1 and ω_2 depend on ψ, according to Eq. (9.6). (We chose to consider the Euler-Lagrange equation in ψ rather than ϕ or θ because of these simplifications.)

Plugging in the partial derivatives, including

$$\frac{\partial \omega_3}{\partial \dot{\psi}} = 1 \quad \frac{\partial \omega_1}{\partial \psi} = \omega_2 \quad \frac{\partial \omega_2}{\partial \psi} = -\omega_1$$

from Eq. (9.6), the Euler-Lagrange equation in ψ becomes

$$\mathcal{I}_{33}\dot{\omega}_3 = \mathcal{I}_{11}\omega_1\omega_2 - \mathcal{I}_{22}\omega_2\omega_1.$$

If an external generalized force Q_ψ in the ψ direction were present, one would add Q_ψ to the right hand side. Since the generalized coordinate ψ is an angle, and $\dot{\psi}$ is a rotation about the $3''' = 3$ axis, Q_ψ is simply the component N_3 of the external torque on the body. Adding this term,

$$\begin{aligned}
\mathcal{I}_{33}\dot{\omega}_3 - (\mathcal{I}_{11} - \mathcal{I}_{22})\omega_1\omega_2 &= N_3 \\
\mathcal{I}_{11}\dot{\omega}_1 - (\mathcal{I}_{22} - \mathcal{I}_{33})\omega_2\omega_3 &= N_1 \\
\mathcal{I}_{22}\dot{\omega}_2 - (\mathcal{I}_{33} - \mathcal{I}_{11})\omega_3\omega_1 &= N_2.
\end{aligned} \tag{9.22}$$

Here, recognizing that the $\mathbf{e}_3$ direction is not unique, we have cyclically permuted the indices to obtain all three equations. These are the famous *Euler equations* prescribing the evolution of the angular velocity of a rigid body. Although we derived them by considering the Euler-Lagrange equation in the Euler angle ψ, Euler's equations involve only the Cartesian components of the angular velocity. These equations can also be derived using only Newtonian mechanics.

10.2. Torque-free symmetrical top.

Consider a rotating body that is *symmetric* ($\mathcal{I}_{11} = \mathcal{I}_{22} \equiv I_0$) and that is free of external torques ($\mathbf{N} = 0$). The first condition is commonly found, as it is satisfied by any object that is cylindrically symmetric about the $\mathbf{e}_3$ axis. However, the latter condition is not often encountered in everyday experience. For the torque to vanish, the top must be located in a "weightless" environment, or be supported on bearings ("gimbals") whose axes intersect its C.M. The earth is one example. As usual, $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the principal axes. Two cases are distinguished. If $\mathcal{I}_{33} \equiv I_3 < I_0$, as would be the case for a cigar, the top is called *prolate*. Otherwise, like a pancake, the top is *oblate*.

For either the prolate or oblate top, Euler's equations reduce to

$$\begin{aligned} I_3\dot\omega_3 = 0 \quad \omega_3 &= \text{constant} \\ \dot\omega_1 &= -\big((I_3/I_0) - 1\big)\omega_2\omega_3 \\ \dot\omega_2 &= \big((I_3/I_0) - 1\big)\omega_3\omega_1 \\ \mathbf{e}_1\dot\omega_1 + \mathbf{e}_2\dot\omega_2 &= \big((I_3/I_0) - 1\big)\omega_3(\omega_1\mathbf{e}_2 - \omega_2\mathbf{e}_1) \\ \frac{d\vec\omega}{dt} &= \vec\Omega \times \vec\omega \\ \vec\Omega &\equiv \mathbf{e}_3\omega_3\big((I_3/I_0) - 1\big). \end{aligned} \tag{9.23}$$

The next to last line describes *precession* of $\vec\omega$ about the $\mathbf{e}_3$ axis with angular velocity $\Omega = \omega_3((I_3/I_0) - 1)$. The precession vanishes if all three principal moments of inertia are equal. Otherwise it is CCW for an oblate top ($I_3 > I_0$), and CW for a prolate top.

While this precession of $\vec\omega$ is fairly straightforward to describe in the body system, the actual motion is complex and quite different from e.g. the familiar precession caused by gravity acting on a top with one point fixed. As viewed in the body system, what is precessing is not the top, but rather the axis about which it is instantaneously rotating. The angular velocity Ω of precession is small compared to ω itself if I_3 and I_0 are nearly equal.

As an example, imagine that the earth is a perfectly rigid body with a slight bulge at the equator, making it oblate. Here the "equator" is defined not by the earth's instantaneous axis of rotation, but rather by a line painted around the earth's circumference at its maximum bulge. Suppose further that, as the result of some cosmic accident, the earth is rotating about an axis at 40° north latitude relative to the equator, e.g. near Denver. Then, as seen by an observer on the earth, this axis of rotation slowly moves east at the same latitude, through Philadelphia, Madrid, etc.

As seen by an observer in the space (inertial) system, the earth's motion is more complicated. This is because the change of $\hat\omega$ is slow relative only to the body axes. As observed in the space system, the axis of the earth's rotation changes more rapidly, because the body axes themselves are spinning with angular frequency ω.

Some sense can be made of the motion observed in the inertial system by considering the angular momentum $\mathbf{L}'$, which is conserved in that system owing to the absence of external torques. Since $\vec\omega$ is precessing, its magnitude is constant. The kinetic energy $\frac{1}{2}\vec\omega\cdot\mathbf{L}'$ (see Eq. (9.12)) also is conserved. Therefore the angle between $\mathbf{L}'$ and $\vec\omega$ is fixed.

Temporarily return to the body axes, and consider the plane formed by $\vec\omega$ and $\mathbf{e}_3$. This plane must contain the angular momentum $\mathbf{L}$ because the body is symmetric ($\mathcal{I}_{11} = \mathcal{I}_{22}$). In the case of an oblate earth, the angular momentum is closer to $\mathbf{e}_3$ than is the angular velocity because I_3 is larger than I_0. Therefore $\mathbf{L}$ lies in the plane *between* $\mathbf{e}_3$ and $\vec\omega$. Both $\vec\omega$ and $\mathbf{L}$ precess CCW about the $\mathbf{e}_3$ axis with the same (small) angular velocity Ω.

Return finally to the space axes. Now $\mathbf{L}'$ is fixed. The other two vectors $\mathbf{e}_3$ and $\vec\omega$ still lie in a plane containing $\mathbf{L}'$, one on either side of $\mathbf{L}'$. Each traces a cone around $\mathbf{L}'$, moving with (large) angular velocity $\approx \omega$. Thus, in the space axes, the earth is observed to *wobble*: it rotates rapidly about an axis which itself is precessing rapidly about $\mathbf{L}'$. This is in addition to the slow precession of $\vec{\omega}$ with respect to $\mathbf{e}_3$.

10.3. Stability of force-free rotation.

In the *symmetric* case just considered, any wobbling arises from an initial misalignment of $\vec{\omega}$ and $\mathbf{e}_3$. When these two vectors are aligned initially, no precession of $\vec{\omega}$ occurs.

Here, for an *asymmetric* torque-free rigid body with unequal principal moments of inertia, $I_1 \neq I_2 \neq I_3$, we investigate the stability of rotation along a direction close to that of a principal axis, say $\mathbf{e}_3$. In other words, $\omega_3 \gg \omega_1$ or ω_2. Euler's equations yield

$$\begin{aligned}
I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 \\
I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 \\
I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 \\
I_1\ddot{\omega}_1 &= (I_2 - I_3)(\dot{\omega}_2\omega_3 + \dot{\omega}_3\omega_2) \\
&= (I_2 - I_3)\big(I_2^{-1}(I_3 - I_1)\omega_3^2\omega_1 - \qquad (9.24)\\
&\quad - I_3^{-1}(I_2 - I_1)\omega_2^2\omega_1\big) \\
&\approx (I_2 - I_3)I_2^{-1}(I_3 - I_1)\omega_3^2\omega_1 \\
0 &= \ddot{\omega}_1 + \frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}\omega_3^2\omega_1.
\end{aligned}$$

The first equality guarantees that ω_3 is nearly constant, since $\omega_1\omega_2$ is assumed to be small. Then the last equality is nearly a harmonic oscillator equation for ω_1, provided that the coefficient of ω_1 in the last term is positive. This occurs if I_3 is either the largest or the smallest principal moment. But if I_3 is intermediate between I_1 and I_2, the coefficient is negative. Then the solution to Eq. (9.24) is an exponentially growing ω_1, and the rotation is unstable.

11. Heavy symmetric top with one point fixed.

11.1. Constants of the motion.

We pass now from the torque-free top to analysis of a "heavy" top which is influenced both by the force of gravity $m\mathbf{g}$ and by forces exerted upon it to keep its pivot point fixed. The top possesses the same symmetry $\mathcal{I}_{11} = \mathcal{I}_{22}$ as considered previously. Instead of the C.M., here we choose the top's pivot point as the origin, so that its kinetic energy can be considered to be purely rotational. Since the pivot point, like the C.M., lies on the $\mathbf{e}_3$ axis, the parallel axis theorem guarantees that the principal moments $I_1 = I_2 \equiv I$ about that point are also equal.

The Euler angles ϕ, θ, and ψ are ideal for describing the orientation of the top. As usual, $\mathbf{e}'_1$, $\mathbf{e}'_2$, and $\mathbf{e}'_3 \equiv -\hat{\mathbf{g}}$ are the fixed axes, with their origin at the pivot point; $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the body axes, with the same origin. According to Euler's convention, the transformation from the space to the body axes consists first of a rotation about $\mathbf{e}'_3$ by ϕ; next a rotation about the line of nodes (the temporary $\mathbf{e}_1$ direction) by θ; and finally a rotation about the $\mathbf{e}_3$ direction by ψ. Therefore ϕ is the azimuth of the top's axis, as viewed in the space system; θ is the polar angle of that axis measured from $\mathbf{e}'_3$; and ψ is the azimuth of the top about the same axis. In other words, spinning of the top about its own axis is represented by $\dot{\psi}$; precession of the top's axis about $\mathbf{e}'_3$ is represented by $\dot{\phi}$; and *nutation*, the (bobbing) variation of the polar angle of the top's axis, is represented by $\dot{\theta}$.

To write the Lagrangian we must express the top's kinetic energy in terms of the Euler angles. Using Eq. (9.6), we evaluate

$$\begin{aligned}
\omega_1^2 + \omega_2^2 &= (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)^2 + \\
&\quad + (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)^2 \\
&= \dot{\phi}^2\sin^2\theta + \dot{\theta}^2 \\
\omega_3^2 &= (\dot{\phi}\cos\theta + \dot{\psi})^2.
\end{aligned}$$

Using these relations, the Lagrangian is

$$\begin{aligned}
\mathcal{L} &= \tfrac{1}{2}(\mathcal{I}_{11}\omega_1^2 + \mathcal{I}_{22}\omega_2^2 + \mathcal{I}_{33}\omega_3^2) - U \\
&= \tfrac{1}{2}I(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \qquad (11.1)\\
&\quad + \tfrac{1}{2}I_3(\dot{\phi}\cos\theta + \dot{\psi})^2 - mgh\cos\theta,
\end{aligned}$$

where h is the distance along the $\mathbf{e}_3$ axis from the pivot point to the C.M.

The Lagrangian is independent of the two cyclic coordinates ϕ and ψ. The corresponding conjugate momenta are constants of the motion:

$$\begin{aligned} p_\phi &\equiv \frac{\partial \mathcal{L}}{\partial \dot\phi} = \text{constant} \\ &= I\dot\phi \sin^2\theta + I_3(\dot\phi\cos\theta + \dot\psi)\cos\theta \\ p_\psi &\equiv \frac{\partial \mathcal{L}}{\partial \dot\psi} = \text{constant} \\ &= I_3(\dot\phi\cos\theta + \dot\psi). \end{aligned} \tag{11.2}$$

Equation (11.2) can be rearranged to express $\dot\phi$ in terms of p_ϕ and p_ψ:

$$\begin{aligned} p_\phi &= I\dot\phi\sin^2\theta + p_\psi\cos\theta \\ \dot\phi &= \frac{p_\phi - p_\psi\cos\theta}{I\sin^2\theta}. \end{aligned} \tag{11.3}$$

The final constant of the motion is obtained by noting that the Lagrangian has no explicit time dependence, so that the Hamiltonian $\mathcal{H} =$ constant $\equiv E$. Since T is a generalized quadratic function of $\dot\phi$, $\dot\theta$, and $\dot\psi$, $\mathcal{H} = T + U$. Then

$$\begin{aligned} \mathcal{H} &= \tfrac12 I(\dot\phi^2\sin^2\theta + \dot\theta^2) + \\ &\quad + \tfrac12 I_3(\dot\phi\cos\theta + \dot\psi)^2 + mgh\cos\theta \\ E &= \tfrac12 I\dot\phi^2\sin^2\theta + \tfrac12 I\dot\theta^2 + \frac{p_\psi^2}{2I_3} + mgh\cos\theta \\ &= \frac{(p_\phi - p_\psi\cos\theta)^2}{2I\sin^2\theta} + \frac{I\dot\theta^2}{2} + \frac{p_\psi^2}{2I_3} + mgh\cos\theta. \end{aligned} \tag{11.4}$$

In the last two lines we used Eqs. (11.2) and (11.3) to eliminate $\dot\phi$ and $\dot\psi$. Eq. (11.4) is the starting point for further analysis.

11.2. Equation of motion in a single coordinate.

Introducing the renormalized energy E' and effective potential U',

$$\begin{aligned} E' &\equiv E - p_\psi^2/2I_3 = \text{constant} \\ U' &\equiv \frac{(p_\phi - p_\psi\cos\theta)^2}{2I\sin^2\theta} + mgh\cos\theta, \end{aligned} \tag{11.5}$$

Eq. (11.4) takes the simple form

$$E' = \tfrac12 I\dot\theta^2 + U'(\theta). \tag{11.6}$$

This is a differential equation for the single coordinate $\theta(t)$.

Before attempting to solve Eq. (11.6), we recognize that U' is infinite both at $\theta = 0$ and at $\theta = \pi$, unless the special condition $|p_\phi| = |p_\psi|$ is satisfied. Therefore U' must reach a minimum at an intermediate polar angle θ_U. When $\theta = \theta_U$ and $\dot\theta = 0$, E' is minimized. There the top moves with the same uniform precession studied in introductory courses. In relation to the effective potential, this minimum energy solution is reminiscent of the circular orbit in the two-body central force problem. As the energy increases (for fixed p_ϕ and p_ψ), the top *nutates* around θ_U between $\theta_{\min}$ and $\theta_{\max}$. This reminds us of the elliptical orbit in the central force problem, except that the frequency of nutation in general is not an integral multiple of the precession frequency.

In the same way as for two-body central force motion, Eq. (11.6) may be rearranged to yield an integral solution to the motion:

$$\begin{aligned} \tfrac12 I\Big(\frac{d\theta}{dt}\Big)^2 &= E' - U'(\theta) \\ t &= \int d\theta \sqrt{\frac{I}{2\big(E' - U'(\theta)\big)}}. \end{aligned} \tag{11.7}$$

Equation (11.6) is simplified by substituting $u \equiv \cos\theta$:

$$\begin{aligned} \dot u &= -\dot\theta\sin\theta = -\dot\theta\sqrt{1-u^2} \\ E' &= I\frac{\dot u^2}{2(1-u^2)} + \frac{(p_\phi - p_\psi u)^2}{2I(1-u^2)} + mghu \\ I^2\dot u^2 &= 2I(1-u^2)(E' - mghu) - (p_\phi - p_\psi u)^2. \end{aligned} \tag{11.8}$$

Unfortunately, this equation is cubic in u.

Further simplification is achieved by releasing the top's axis from rest: $\dot\theta(0) = \dot\phi(0) = 0$, with $u(0) \equiv u_0 \equiv \cos\theta_0$. Referring to Eq. (11.3), we see that $p_\phi - p_\psi\cos\theta_0$ also vanishes. Then, in Eq. (11.5), so does the first term in U' when $t = 0$. Since $\dot\theta$ also vanishes at that time, Eq. (11.6) reduces to $E' = mghu_0$. Equation (11.8) becomes

$$\begin{aligned} I^2\dot{u}^2 &= 2I(1-u^2)mgh(u_0-u) - (p_\phi - p_\psi u)^2 \\ &= 2I(1-u^2)mgh(u_0-u) - p_\psi^2(u_0-u)^2 \\ &= p_\psi^2\big(\alpha(1-u^2)(u_0-u) - (u_0-u)^2\big) \\ \alpha &\equiv \frac{mgh}{p_\psi^2/2I}. \end{aligned} \tag{11.9}$$

In the last line we have introduced the dimensionless constant α, which is a factor of order unity ($I/2I_3$) multiplied by the ratio of the *range in potential energy* ($2mgh$) to the *initial kinetic energy* ($p_\psi^2/2I_3$).

Although its constants are in neater form, Eq. (11.9) is still a cubic in u. One new piece of information falls out easily. Obviously $\dot{u} = 0$ when $u = u_0$, as the initial conditions demand. In addition, $\dot{u} = 0$ when $u_0 - u = \alpha(1 - u^2)$. Denote this second turning point by $u = u_0 - \Delta u$. The condition for $\dot{u} = 0$ becomes

$$\begin{aligned} \Delta u &= \alpha\big(1 - (u_0 - \Delta u)^2\big) \\ \Delta u &= \alpha(1 - u_0^2 + 2u_0\Delta u - \Delta u^2). \end{aligned} \tag{11.10}$$

The range Δu within which the top nutates can be obtained by solving this quadratic equation.

11.3. Nutation of a fast top.

Another advantage of Eq. (11.9), relative to earlier versions, is that the right hand side contains two distinct terms. Depending on the value of α, it may be possible to neglect one with respect to the other. For example, a *fast top* has an initial kinetic energy which greatly exceeds its range in potential energy: $\alpha \ll 1$. Then Eq. (11.10) requires Δu to be small, simplifying to

$$\Delta u \approx \alpha(1 - u_0^2) = \alpha \sin^2\theta_0. \tag{11.11}$$

To solve Eq. (11.9) for the special case of the fast top, we apply the method of perturbations. Since u is known to vary between u_0 and $u_0 - \Delta u = u_0 - \alpha\sin^2\theta_0$, we expand u about its central value $u_0 - \frac{1}{2}\Delta u$:

$$\begin{aligned} u &= u_0 - \tfrac{1}{2}\Delta u + \delta \\ &= u_0 - \tfrac{\alpha}{2}\sin^2\theta_0 + \delta. \end{aligned} \tag{11.12}$$

Here the perturbation δ is of the same small order as α.

Now we insert Eq. (11.12) into Eq. (11.9) and retain terms to second order in α and δ:

$$\begin{aligned} &\frac{I^2}{p_\psi^2}\dot{u}^2 = \alpha(1-u^2)(u_0-u) - (u_0-u)^2 \\ &= (u_0-u)\big(\alpha(1-u^2) - (u_0-u)\big) \\ &= (\tfrac{\alpha}{2}\sin^2\theta_0 - \delta)\big(\alpha(1-u^2) - (\tfrac{\alpha}{2}\sin^2\theta_0 - \delta)\big) \\ &\approx (\tfrac{\alpha}{2}\sin^2\theta_0 - \delta)\big(\alpha\sin^2\theta_0 - (\tfrac{\alpha}{2}\sin^2\theta_0 - \delta)\big) \\ &= (\tfrac{\alpha}{2}\sin^2\theta_0 - \delta)(\tfrac{\alpha}{2}\sin^2\theta_0 + \delta) \\ &\frac{I^2}{p_\psi^2}\dot{\delta}^2 = \frac{\alpha^2}{4}\sin^4\theta_0 - \delta^2. \end{aligned} \tag{11.13}$$

Equation (11.13) is merely quadratic in δ and may be solved by taking the time derivative:

$$\begin{aligned} 2\dot{\delta}\ddot{\delta} &= -\frac{p_\psi^2}{I^2}2\delta\dot{\delta} \\ \ddot{\delta} &= -\frac{p_\psi^2}{I^2}\delta \\ \Omega_{\text{nutation}} &= \frac{p_\psi}{I}. \end{aligned} \tag{11.14}$$

For comparison, the average angular velocity of precession is:

$$\begin{aligned} \langle\dot{\phi}\rangle &= \Big\langle\frac{p_\phi - p_\psi\cos\theta}{I\sin^2\theta}\Big\rangle \\ &= \Big\langle\frac{p_\psi(u_0-u)}{I\sin^2\theta}\Big\rangle \\ &\approx \Big\langle\frac{p_\psi\Delta u}{2I\sin^2\theta}\Big\rangle \\ &\approx \frac{p_\psi\alpha\sin^2\theta_0}{2I\sin^2\theta_0} = \frac{\alpha}{2}\frac{p_\psi}{I}. \end{aligned} \tag{11.15}$$

Therefore, independently of θ_0, the fast top nutates $\Omega_{\text{nutation}}/\langle\dot{\phi}\rangle = 2/\alpha$ times for each period of precession.

11.4. Stability of a sleeping top.

Retaining the same initial conditions, we consider finally the special case of an initially upright ("sleeping") top, i.e. $\theta_0 = 0$. Because of this additional simplification, it will no longer be necessary to assume that the top is *fast*, i.e. that $\alpha \ll 1$. Beginning with the first equality in Eq. (11.13), set $u_0 = 1$ and define $\epsilon \equiv 1 - u$:

$$\begin{aligned}
\frac{I^2}{p_\psi^2}\dot{u}^2 &= \alpha(1-u^2)(u_0-u) - (u_0-u)^2 \\
&= \alpha(1-u^2)(1-u) - (1-u)^2 \\
&= (1-u)^2\big(\alpha(1+u) - 1\big) \\
\frac{I^2}{p_\psi^2}\dot{\epsilon}^2 &= \epsilon^2\big(\alpha(2-\epsilon) - 1\big).
\end{aligned} \tag{11.16}$$

Differentiating both sides with respect to time,

$$\begin{aligned}
\frac{I^2}{p_\psi^2}\dot{\epsilon}\ddot{\epsilon} &= \epsilon\dot{\epsilon}\big(\alpha(2-\epsilon) - 1\big) - \epsilon^2\alpha\dot{\epsilon} \\
\frac{I^2}{p_\psi^2}\ddot{\epsilon} &= \epsilon\big(\alpha(2-\epsilon) - 1\big) - \epsilon^2\alpha \\
&= \epsilon\big(2\alpha(1-\epsilon) - 1\big) \\
&\approx -(1-2\alpha)\epsilon.
\end{aligned} \tag{11.17}$$

When $\alpha < \frac{1}{2}$, perturbations about $\cos\theta = 1$ oscillate stably with angular frequency

$$\Omega_{\text{nutation}} = \frac{p_\psi}{I}\sqrt{1-2\alpha}. \tag{11.18}$$

When $\alpha \ll 1$, this reduces to Eq. (11.14). When $\alpha > \frac{1}{2}$, the sleeping top is unstable. This is a familiar observation. As friction slows $\dot{\psi}$ and raises α, an initially upright top suddenly nutates violently.

12. Coupled oscillatory motion.

12.1. Lagrangian for small oscillations about equilibrium.

Our approach to the *coupled oscillator problem* will be to develop the most general method, usable for any number n of generalized coordinates q_k, $1 \le k \le n$. Here k runs over *both* the number of *dimensions* and the number of *particles* in the problem; for example, a problem with two particles in three dimensions has $n = 6$.

For a velocity-independent potential $U(q_k)$, about a local minimum $U_{\min}$ where all the q's are defined to vanish, we may expand to second order

$$\begin{aligned}
U - U_{\min} &= \sum_k q_k \frac{\partial U}{\partial q_k}\Big|_{U_{\min}} + \\
&\quad + \tfrac{1}{2}\sum_{kl} q_k q_l \frac{\partial^2 U}{\partial q_k \partial q_l}\Big|_{U_{\min}} \\
&\equiv 0 + \tfrac{1}{2}\sum_{kl} q_k q_l \mathcal{K}_{kl} \\
&\equiv \tfrac{1}{2} q_k \mathcal{K}_{kl} q_l \\
&\equiv \tfrac{1}{2}\tilde{q}^t \mathcal{K}\tilde{q} \\
\mathcal{K}_{kl} &\equiv \frac{\partial^2 U}{\partial q_k \partial q_l}\Big|_{U_{\min}}.
\end{aligned} \tag{12.1}$$

In the second equality we used the fact that all derivatives with respect to the q_k vanish at the local minimum. Notice the summation over k and l implied by the repeated indices in the third equality, and the matrix notation in the fourth.

The elements $\mathcal{K}_{kl}$ of the *spring constant matrix* $\mathcal{K}$ are *constants*. Likewise, we assume that the potential energy T for this system can be written in terms of the constant *mass matrix* $\mathcal{M}$:

$$\begin{aligned}
T &\equiv \tfrac{1}{2}\sum_{kl} \dot{q}_k \dot{q}_l \mathcal{M}_{kl} \\
&\equiv \tfrac{1}{2}\dot{q}_k \mathcal{M}_{kl} \dot{q}_l.
\end{aligned} \tag{12.2}$$

This is a much stronger assumption than was made in order to obtain Eq. (12.1). For example, the coordinates must be Cartesian; otherwise the elements $\mathcal{M}_{kl}$ of the mass matrix would be functions of the q_k. Then the Lagrangian is

$$\mathcal{L}(\{q_i\},\{\dot{q}_i\}) = \tfrac{1}{2}\dot{q}_k \mathcal{M}_{kl}\dot{q}_l - \tfrac{1}{2} q_k \mathcal{K}_{kl} q_l. \tag{12.3}$$

The oscillator is *coupled* if *any* of the off-diagonal elements of $\mathcal{K}$ *or* $\mathcal{M}$ are nonzero. Otherwise, the Lagrangian merely describes n *uncoupled* oscillators, each affected by none of the others.

Since the matrix elements in the Lagrangian are constant, the Euler-Lagrange equations in each of the n coordinates q_k yield n different equations of the form

$$\mathcal{K}_{kl} q_l + \mathcal{M}_{kl}\ddot{q}_l = 0, \tag{12.4}$$

each with $2n$ terms. In obtaining Eq. (12.4), one makes use of the fact that $\mathcal{K}$ is *symmetric* by definition, and that we *force* $\mathcal{M}$ to be symmetric by associating equal amounts of kinetic energy with $\mathcal{M}_{kl}$ and $\mathcal{M}_{lk}$.

12.2. Harmonic solution to Euler-Lagrange equation.

In analogy with simple harmonic motion of the undamped oscillator, we speculate that there exist solutions to Eq. (12.4) for which each coordinate q_k executes harmonic motion at the *same frequency* and with the *same phase*, give or take π, as every other coordinate. We *do* allow the amplitude of oscillation a_k to vary with k. Such solutions exist; they are called *normal modes of oscillation*. By analogy with the complex exponential method, taking a_k to be a real constant, substitute

$$q_k(t) = \Re\big(a_k e^{i(\omega t+\delta)}\big) \tag{12.5}$$

in Eq. (12.4):

$$\Re\big((\mathcal{K}_{kl} - \omega^2\mathcal{M}_{kl})a_l e^{i(\omega t+\delta)}\big) = 0. \tag{12.6}$$

As usual, we choose to solve the complex equation of which Eq. (12.6) is the real part. Factoring out the common phase,

$$\begin{aligned}(\mathcal{K}_{kl} - \omega^2\mathcal{M}_{kl})a_l &= 0\\ (\mathcal{K} - \omega^2\mathcal{M})\tilde{a} &= 0,\end{aligned} \tag{12.7}$$

where $\tilde{a}$ is a column vector with elements equal to the a_l's. Since everything else in the linear Eq. (12.7) is real, the a_l's may be defined to be real as well. This supports our original speculation.

12.3. Normal mode eigenvalue problem.

Equation (12.7) is similar to the eigenvalue problem encountered when we diagonalized the inertia tensor. In fact, since both $\mathcal{K}$ and $\mathcal{M}$ are real symmetric matrices, it is the *same* problem, except that $\mathcal{K} - \omega^2\mathcal{M}$ replaces $\mathcal{I} - e\mathrm{I}$. By the same arguments used in section (9.6), there exist n *normal frequencies*^{2} ω_r^2, $1 \le r \le n$. These correspond to the real positive eigenvalues of the inertia tensor. Like those eigenvalues, some of the normal frequencies may be the same (these are called "degenerate"). Corresponding to the eigenvectors of the inertia tensor, there exist n real *normal mode vectors* $\tilde{a}^r$.

The normal mode vectors are orthogonal, but only in a special way:

$$\begin{aligned}(\tilde{a}^r)^t\tilde{a}^s &\neq 0 \text{ when } r \neq s, \text{ but}\\ (\tilde{a}^r)^t\mathcal{M}\tilde{a}^s &= 0 \text{ when } r \neq s.\end{aligned} \tag{12.8}$$

This special type of orthogonality is readily understood in the context of the proof in section (9.6). That proof used the fact that the eigenvalue e multiplies the unit matrix I in the eigenvalue equation (9.16). However, in the coupled oscillator problem, the normal frequency^2 ω^2 multiplies $\mathcal{M}$ rather than I.

Equation (12.8) still leaves us free to choose the *lengths* of the normal mode vectors. Our choice produces the most elegant condition:

$$(\tilde{a}^r)^t\mathcal{M}\tilde{a}^s = \delta_{rs}. \tag{12.9}$$

The normal mode vectors are said to be *orthonormal in a space having the metric* $\mathcal{M}$, rather than in a space having the usual unit metric I.

Although the analogy with the inertia tensor eigenvalue problem is complete, we record here the steps required to obtain the normal frequencies and the normal mode vectors. First we identify the spring constant and mass matrices for the particular problem at hand. Then, for a solution to Eq. (12.7) to exist, we demand that the *secular equation* be satisfied:

$$\det(\mathcal{K} - \omega^2\mathcal{M}) = 0. \tag{12.10}$$

This is an n^{th} order polynomial equation for ω^2. Its n roots are the normal frequencies^2 ω_r^2. Next, for *each* of these n frequencies, we solve n simultaneous equations of the form

$$(\mathcal{K} - \omega_r^2\mathcal{M})\tilde{a}^r = 0 \tag{12.11}$$

to obtain each of the normal mode vectors $\tilde{a}^r$. Finally, we adjust the lengths of each of the $\tilde{a}^r$'s in order to satisfy the orthonormality requirement (12.9).

12.4. Transformation to normal coordinates.

In our study of the inertia tensor $\mathcal{I}$, we benefited by considering the square matrix Λ^t whose columns are its eigenvectors. The untransposed version Λ of that matrix was found to represent the orthogonal transformation that diagonalizes $\mathcal{I}$ by means of the *similarity transformation* $\Lambda\mathcal{I}\Lambda^t$.

Correspondingly, we now form the matrix $\mathcal{A}$ whose columns are the normal mode vectors. This means that the $(kr)^{\text{th}}$ element of $\mathcal{A}$ is the k^{th} component of the r^{th} normal mode vector:

$$\mathcal{A}_{kr} \equiv a_k^r. \tag{12.12}$$

The benefits of considering $\mathcal{A}$ are even more striking. The orthonormality condition (12.9) becomes still more elegant:

$$\begin{aligned} \delta_{rs} &= (\tilde{a}^r)^t \mathcal{M} \tilde{a}^s \\ &= a_k^r \mathcal{M}_{kl} a_l^s \\ &= \mathcal{A}_{kr} \mathcal{M}_{kl} \mathcal{A}_{ls} \\ &= \mathcal{A}_{rk}^t \mathcal{M}_{kl} \mathcal{A}_{ls} \\ &= (\mathcal{A}^t \mathcal{M} \mathcal{A})_{rs} \\ \mathrm{I} &= \mathcal{A}^t \mathcal{M} \mathcal{A}. \end{aligned} \tag{12.13}$$

Equation (12.13) states that, by means of the *congruence transformation* $\mathcal{A}^t\mathcal{M}\mathcal{A}$, the matrix $\mathcal{A}$ of normal mode vectors not only diagonalizes the mass matrix $\mathcal{M}$; it reduces $\mathcal{M}$ to the unit matrix I.

A similar proof using the spring constant matrix yields

$$\begin{aligned} \mathcal{A}^t \mathcal{K} \mathcal{A} &\equiv \Omega^2 \\ &= \begin{pmatrix} \omega_1^2 & 0 & \cdots & 0 \\ 0 & \omega_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_n^2 \end{pmatrix}. \end{aligned} \tag{12.14}$$

Again, by means of the congruence transformation $\mathcal{A}^t\mathcal{K}\mathcal{A}$, the matrix $\mathcal{A}$ of normal mode vectors diagonalizes the spring constant matrix $\mathcal{K}$. The resulting matrix Ω^2 is the *diagonal matrix of normal frequencies*[2] ω_r^2.

How is it possible that the same transformation $\mathcal{A}$ is able to diagonalize two different real symmetric matrices $\mathcal{M}$ and $\mathcal{K}$? Imagine first performing a simple rotation to diagonalize $\mathcal{M}$. Then transform to a renormalized set of generalized coordinates so that $\mathcal{M}$ is proportional to the unit matrix. Finally, perform a second rotation to diagonalize $\mathcal{K}$. This second rotation preserves the diagonal form of $\mathcal{M}$ because it is already the unit matrix.

So far we have been discussing the effect of $\mathcal{A}$ upon the mass and spring constant matrices. Now consider using $\mathcal{A}$ to transform *from* a new set of n generalized coordinates $\{Q_r\}$ *to* the original set of n generalized coordinates $\{q_k\}$:

$$\begin{aligned} q_k(t) &\equiv \sum_r \mathcal{A}_{kr} Q_r(t) \\ \tilde{q} &\equiv \mathcal{A}\tilde{Q} \\ \tilde{Q} &= \mathcal{A}^{-1}\tilde{q} \\ &= \mathcal{A}^t \mathcal{M} \tilde{q}. \end{aligned} \tag{12.15}$$

In the last line we set $\mathcal{A}^{-1} = \mathcal{A}^t\mathcal{M}$ using Eq. (12.13). The $\{Q_r\}$ are called *normal coordinates*.

In the original basis $\{q_k\}$, neither $\mathcal{M}$ nor $\mathcal{K}$ were diagonal. But in the new basis $\{Q_r\}$,

$$\begin{aligned} T &= \tfrac{1}{2}\dot{\tilde{q}}^t \mathcal{M} \dot{\tilde{q}} \\ &= \tfrac{1}{2}(\mathcal{A}\dot{\tilde{Q}})^t \mathcal{M} \mathcal{A} \dot{\tilde{Q}} \\ &= \tfrac{1}{2}\dot{\tilde{Q}}^t \mathcal{A}^t \mathcal{M} \mathcal{A} \dot{\tilde{Q}} \\ &= \tfrac{1}{2}\dot{\tilde{Q}}^t \mathrm{I} \dot{\tilde{Q}} \\ &= \tfrac{1}{2}\dot{\tilde{Q}}^t \dot{\tilde{Q}}. \end{aligned} \tag{12.16}$$

Similarly,

$$U = \tfrac{1}{2}\tilde{Q}^t \Omega^2 \tilde{Q}. \tag{12.17}$$

In the new basis $\{Q_r\}$, called the *normal basis*, both the mass and spring constant matrices are diagonal. Application of the Euler-Lagrange equations yields one simple harmonic equation for each normal coordinate. In the new basis, Eq. (12.4) is simply

$$\omega_r^2 Q_r + \ddot{Q}_r = 0. \tag{12.18}$$

The oscillation of each normal coordinate is *totally decoupled* from that of any other normal coordinate. The normal modes of oscillation are connected only through the initial conditions. Thereafter, oblivious to the others, each normal mode forever rattles away at its own normal frequency.

When a system of oscillators is excited in *only one* normal mode, what are the motions of the *original* generalized coordinates q_k? The simple answer is provided by Eq. (12.15). For example, if only mode seven is excited, only Q_7 is nonzero:

$$q_k(t) = A_{k7} Q_7(t) = a_k^7 Q_7(t). \tag{12.19}$$

The relative amplitude of the motion of each of the n generalized coordinates is given simply by the relative size of the n components a_k^7 of the seventh normal mode vector. All coordinates q_k execute simple harmonic motion with the same angular frequency ω_7 and the same phase.

Of course, the general solution involves the excitation of all n normal modes. Then each original generalized coordinate executes the sum of n simple harmonic oscillations with n (generally) different angular frequencies and phases. For any coordinate, the amplitude of a particular oscillation frequency is given by the relative amplitude of motion of that coordinate for the particular normal mode which has that frequency, multiplied by the amplitude with which that particular normal mode is excited, as determined by the initial conditions.

12.5. Obtaining normal mode amplitudes from initial conditions.

Let's express the previous paragraph in equations rather than words. The initial conditions require each normal coordinate $Q_r(t)$ to have a unique amplitude and phase:

$$Q_r(t) = \Re(P_r e^{i\omega_r t}), \tag{12.20}$$

where P_r is a complex constant. From Eq. 12.15, the original generalized coordinates q_k become

$$q_k(t) = \Re\Big(\sum_r \mathcal{A}_{kr} P_r e^{i\omega_r t}\Big). \tag{12.21}$$

The two constants $\mathcal{A}_{kr}$ and P_r are different in form and in function. $\mathcal{A}_{kr}$ is *real*, with a magnitude *fixed* by the orthonormality condition (12.9). For oscillation in the r^{th} normal mode, it determines the relative amplitude of the k^{th} generalized coordinate q_k. P_r is *complex*, with a magnitude and phase *adjusted* to fit the initial conditions. It *is* the amplitude with which the r^{th} normal mode is excited.

As for the initial conditions, suppose that

$$q_k(0) \equiv q_{0k} \ ; \ \dot{q}_k(0) \equiv \dot{q}_{0k}, \tag{12.22}$$

where q_{0k} and $\dot{q}_{0k}$ are real constants. (Note that $\dot{q}_{0k} \neq dq_{0k}/dt$!) Suppose also that the complex normal mode amplitude P_r is expressed in terms of its real and imaginary parts:

$$P_r \equiv R_r + \frac{S_r}{i\omega_r}, \tag{12.23}$$

where R_r and S_r are real constants. Then from Eqs. (12.21) and (12.22),

$$\begin{aligned} q_{0k} &= \sum_r \mathcal{A}_{kr} R_r \\ \dot{q}_{0k} &= \sum_r \mathcal{A}_{kr} S_r. \end{aligned} \tag{12.24}$$

Transforming to matrix notation,

$$\begin{aligned} \tilde{q}_0 &= \mathcal{A}\tilde{R}; \quad \tilde{R} = \mathcal{A}^{-1}\tilde{q}_0 \\ \tilde{\dot{q}}_0 &= \mathcal{A}\tilde{S}; \quad \tilde{S} = \mathcal{A}^{-1}\tilde{\dot{q}}_0. \end{aligned} \tag{12.25}$$

Using $\mathcal{A}^{-1} = \mathcal{A}^t \mathcal{M}$, we have finally

$$\begin{aligned} \tilde{R} &= \mathcal{A}^t \mathcal{M} \tilde{q}_0 \\ \tilde{S} &= \mathcal{A}^t \mathcal{M} \tilde{\dot{q}}_0. \end{aligned} \tag{12.26}$$

With $\tilde{R}$ and $\tilde{S}$ specified, so is $\tilde{P}$ from Eq. (12.23), yielding the $q_k(t)$ from Eq. (12.21).

12.6. Energy in normal modes.

Using Eqs. (12.16), (12.17), (12.20), and (12.23), the conserved total energy is

$$\begin{aligned} E &= T + U \\ &= \tfrac{1}{2}\dot{\tilde{Q}}^t(t)\dot{\tilde{Q}}(t) + \tfrac{1}{2}\tilde{Q}^t(t)\Omega^2\tilde{Q}(t) \\ &= \tfrac{1}{2}\dot{\tilde{Q}}^t(0)\dot{\tilde{Q}}(0) + \tfrac{1}{2}\tilde{Q}^t(0)\Omega^2\tilde{Q}(0) \\ &= \tfrac{1}{2}\tilde{S}^t\tilde{S} + \tfrac{1}{2}\tilde{R}^t\Omega^2\tilde{R} \\ &= \tfrac{1}{2}\sum_r (S_r^2 + \omega_r^2 R_r^2) \\ &= \tfrac{1}{2}\sum_r \omega_r^2 P_r^* P_r \\ &= \tfrac{1}{2}\tilde{P}^\dagger\Omega^2\tilde{P}. \end{aligned} \qquad (12.27)$$

Although energy *within* a particular normal mode is shifted back and forth between kinetic and potential forms, as occurs in any harmonic oscillator, the sum of potential and kinetic energies in that mode is constant. According to Eq. (12.27), the total energy E is just the sum of energies in the individual modes, which in turn depend only upon the normal frequencies ω_r and the moduli $|P_r|$ of the mode amplitudes.

12.7. Driven coupled oscillator.

So far we have been discussing the general solution to the homogeneous equation, corresponding to free vibrations. If a driving term is present, we need a particular solution of the inhomogenous equation to add to the homogeneous solution. Suppose that each of the original generalized coordinates q_k is subjected to a generalized force

$$f_k \equiv \Re(g_k e^{i\omega t}), \qquad (12.28)$$

where g_k is a complex constant. Note that the driving frequency ω is assumed to be the same for all coordinates.

We shall express the generalized force F_r acting on the normal coordinate Q_r as

$$F_r \equiv \Re(G_r e^{i\omega t}). \qquad (12.29)$$

Then, recalling from Eq. (12.15) that $q_k = \mathcal{A}_{kr}Q_r$, we require that the work done by the force when calculated in either coordinate system be the same:

$$\begin{aligned} \sum_r F_r\, dQ_r &= \sum_k f_k\, dq_k \\ &= \sum_k f_k \sum_r \mathcal{A}_{kr}\, dQ_r \\ &= \sum_r \big(\sum_k \mathcal{A}_{kr} f_k\big) dQ_r \\ F_r &= \sum_k \mathcal{A}_{kr} f_k \\ \tilde{F} &= \mathcal{A}^t \tilde{f} \\ \tilde{G} &= \mathcal{A}^t \tilde{g}. \end{aligned} \qquad (12.30)$$

With the addition of the generalized force, the Euler-Lagrange equation satisfied by the normal coordinate Q_r becomes

$$\ddot{Q}_r = -\omega_r^2 Q_r + \Re(G_r e^{i\omega t}). \qquad (12.31)$$

Seeking a solution of the form

$$Q_r(t) \equiv \Re(P'_r e^{i\omega t}), \qquad (12.32)$$

where P'_r is a complex constant, we easily find

$$P'_r = \frac{G_r}{\omega_r^2 - \omega^2}. \qquad (12.33)$$

Introducing

$$\mathcal{T}^2 \equiv \begin{pmatrix} \frac{1}{\omega_1^2-\omega^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{\omega_2^2-\omega^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\omega_n^2-\omega^2} \end{pmatrix}, \qquad (12.34)$$

the particular solution is

$$\begin{aligned} \tilde{P}' &= \mathcal{T}^2\tilde{G} \\ &= \mathcal{T}^2\mathcal{A}^t\tilde{g} \\ \mathcal{A}\tilde{P}' &= \mathcal{A}\mathcal{T}^2\mathcal{A}^t\tilde{g} \\ \Re(\mathcal{A}\tilde{P}' e^{i\omega t}) &= \Re(\mathcal{A}\mathcal{T}^2\mathcal{A}^t\tilde{g}e^{i\omega t}) \\ \mathcal{A}\tilde{Q}(t) &= \mathcal{A}\mathcal{T}^2\mathcal{A}^t\Re(\tilde{g}e^{i\omega t}) \\ \tilde{q}(t) &= \mathcal{A}\mathcal{T}^2\mathcal{A}^t\tilde{f}(t), \end{aligned} \qquad (12.35)$$

where it is understood that the components of $\tilde{f}(t)$ are sinusoidal in ω. Equation (12.35) expresses a simple result: the steady-state response $\tilde{q}(t)$ of the original generalized coordinates to a generalized sinusoidal driving force $\tilde{f}(t)$ is given by the operator $\mathcal{A}\mathcal{T}^2\mathcal{A}^t$, which is just $\mathcal{T}^2$ after a similarity transformation by $\mathcal{A}$.

13. Lagrangian density for continuous systems.

13.1. Calculus of variations for two independent variables.

In section (4.1) we considered the problem of finding the path $y(t)$ such that the action

$$J = \int_{t_1}^{t_2} dt\, \mathcal{L}(y, \tfrac{\partial y}{\partial t}, t)$$

is extremized, where the limits of integration are fixed, and where $\mathcal{L}$ (later called the Lagrangian) is a function of the indicated variables. The solution was given by the Euler equation (4.3):

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial(\frac{\partial y}{\partial t})} = \frac{\partial \mathcal{L}}{\partial y}. \tag{13.1}$$

Here we have chosen to write the first derivative of y with respect to t as $\frac{\partial y}{\partial t}$ rather than as $\dot{y}$. (Since y is a function only of the independent variable t, the partial and total time derivatives of y are equivalent.)

Now we consider a slightly more general problem. The new action to be extremized is a *two*-dimensional integral of a new function $\mathcal{L}'$, later to be called the *Lagrangian density*. The path y which extremizes the action is a function of *two* independent variables s and t. The integral is taken over these same two variables. The Lagrangian density $\mathcal{L}'(y, \frac{\partial y}{\partial s}, \frac{\partial y}{\partial t}, s, t)$ is a function of y and its partial derivatives with respect both to s and t. It may also be a function of s and/or t explicitly. The action is

$$J = \int_{s_1}^{s_2} ds \int_{t_1}^{t_2} dt\, \mathcal{L}'(y, \tfrac{\partial y}{\partial s}, \tfrac{\partial y}{\partial t}, s, t),$$

where the limits of *both* integrals are fixed.

It is not surprising that a derivation of the same type as that in section (4.1) gives a slightly more general Euler equation:

$$\frac{d}{ds}\frac{\partial \mathcal{L}'}{\partial(\frac{\partial y}{\partial s})} + \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial(\frac{\partial y}{\partial t})} = \frac{\partial \mathcal{L}'}{\partial y}. \tag{13.2}$$

This is the same as Eq. (13.1) except for the added first term, which is equivalent to the second term with t replaced by s. The meaning of $\frac{d}{ds}$ and $\frac{d}{dt}$ requires clarification. For example, $\frac{d}{dt}$ is *total* because it includes the variation of $\mathcal{L}'$ with respect to t both *explicitly* and *implicitly* through the dependence upon t of y and its derivatives. However, $\frac{d}{dt}$ is also *partial* because the other independent variable s must be held fixed throughout the differentiation. To be excessively precise,

$$\begin{aligned}\frac{d}{dt} \equiv & \left(\frac{\partial}{\partial t}\right)_{y, \frac{\partial y}{\partial s}, \frac{\partial y}{\partial t}, s} + \\ & + \left(\frac{\partial y}{\partial t}\right)_s \left(\frac{\partial}{\partial y}\right)_{\frac{\partial y}{\partial s}, \frac{\partial y}{\partial t}, s, t} + \\ & + \left(\frac{\partial \frac{\partial y}{\partial s}}{\partial t}\right)_s \left(\frac{\partial}{\partial \frac{\partial y}{\partial s}}\right)_{y, \frac{\partial y}{\partial t}, s, t} + \\ & + \left(\frac{\partial \frac{\partial y}{\partial t}}{\partial t}\right)_s \left(\frac{\partial}{\partial \frac{\partial y}{\partial t}}\right)_{y, \frac{\partial y}{\partial s}, s, t},\end{aligned} \tag{13.3}$$

where the subscripts are the variables to be held fixed during the differentiation. Note that terms similar to those in the first, second, and last lines of this expression would be present even if we had only one independent variable (t).

The generalization of Eq. (13.2) to n new independent variables s_i, $1 \le i \le n$, is obvious:

$$\frac{d}{ds_i}\frac{\partial \mathcal{L}'}{\partial(\frac{\partial y}{\partial s_i})} + \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial(\frac{\partial y}{\partial t})} = \frac{\partial \mathcal{L}'}{\partial y}, \tag{13.4}$$

where, as usual, summation over the repeated index i is implied.

13.2. Hamilton's Principle for continuous systems.

The foregoing section was concerned merely with a mathematical problem in variational calculus. As in section (5.1), the connection with physics is recovered by identifying the independent variable t with the *time*. Of what use is the other independent variable s?

One answer is that s may play the role of a *field variable* in one spatial dimension. For example, to describe the state of a string which may be displaced in one transverse direction, we must specify the displacement y as a function both of time t and of the position s along the string. The fixed limits of integration t_1 and t_2 correspond to the fixed time interval over which the action is to be minimized. Correspondingly, the fixed limits of integration s_1 and s_2 correspond to the *fixed endpoints of the string*.

In order to preserve the action's units, we assign to the integrand $\mathcal{L}'$ the units of the Lagrangian $\mathcal{L} = T - U$ divided by those of s. In our example of a string, $\mathcal{L}'$ is the *Lagrangian per unit length* along the string – that is, $T - U$ per unit length. More generally, $\mathcal{L}'$ is called the *Lagrangian density*.

Hamilton's Principle for continuous systems asserts that the system will evolve along the path $y(s, t)$ which minimizes the action – the integral with respect to s and t of the Lagrangian density $\mathcal{L}'$. This means that y satisfies the Euler-Lagrange equation (13.2). Again, the justification of Hamilton's Principle relies on the fact that it reproduces the solutions to problems amenable to Newtonian analysis, while it yields solutions to more complex problems that are confirmed by experiment.

13.3. Transverse wave equation for a string.

Consider an infinitesimal piece Δs of string in a gravity-free region. We assume that any motion is possible only in the single transverse direction y. The kinetic energy of the piece of string is

$$\Delta T = \tfrac{1}{2}\mu\Big(\frac{\partial y}{\partial t}\Big)^2 \Delta s,$$

where μ is the string mass per unit length.

The potential energy requires a bit more discussion. If the string's slope is $\frac{\partial y}{\partial s}$, according to Pythagoras' theorem the length of the piece of string is increased with respect to its equilibrium length by

$$\begin{aligned}\Delta l &= \sqrt{(\Delta s)^2 + (\Delta y)^2} - \Delta s \\ &= \Big(\sqrt{1 + \big(\frac{\partial y}{\partial s}\big)^2} - 1\Big)\Delta s \\ &\approx \tfrac{1}{2}\Big(\frac{\partial y}{\partial s}\Big)^2 \Delta s,\end{aligned}$$

where the approximation is valid for small slopes. If the string is stretched with tension τ, the incremental potential energy associated with this extra length is $\Delta U = \tau\,\Delta l$.

The Lagrangian density corresponding to ΔT and ΔU is

$$\begin{aligned}\mathcal{L}' &= \frac{\Delta T}{\Delta s} - \frac{\Delta U}{\Delta s} \\ &= \tfrac{1}{2}\mu\Big(\frac{\partial y}{\partial t}\Big)^2 - \tfrac{1}{2}\tau\Big(\frac{\partial y}{\partial s}\Big)^2.\end{aligned} \tag{13.5}$$

Applying the Euler-Lagrange Eq. (13.2) to $\mathcal{L}'$,

$$\begin{aligned}-\frac{d}{ds}\Big(\tau\frac{\partial y}{\partial s}\Big) + \frac{d}{dt}\Big(\mu\frac{\partial y}{\partial t}\Big) &= 0 \\ -\tau\frac{\partial^2 y}{\partial s^2} + \mu\frac{\partial^2 y}{\partial t^2} &= 0 \\ \frac{\partial^2 y}{\partial s^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} &= 0 \\ c &\equiv \sqrt{\frac{\tau}{\mu}}.\end{aligned} \tag{13.6}$$

In the last line we defined the *phase velocity* c. Equation (13.6), the *wave equation*, is the starting point for our discussion of waves in one dimension.

14. Waves.

14.1. General solution to the wave equation.

The general solution to the wave equation (13.6),

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0, \tag{14.1}$$

is

$$y(x,t) = y_+(x-ct) + y_-(x+ct), \tag{14.2}$$

where y_+ and y_- are any twice differentiable functions of their arguments. The first (second) term is an arbitrary smooth shape travelling in the positive (negative) x direction with velocity c. Following the usual convention, in Eq. (14.1) we substituted x for s. Nevertheless, it cannot be emphasized too strongly that x and y play *completely* different roles in the Euler-Lagrange equation (13.2). The former is an *independent* variable like the time; the latter is a variable that *depends* on x and t.

The fact that Eq. (14.2) solves the wave equation is easily verified by substitution. The fact that it is a *general* solution is illustrated by obtaining the $\{y_+, y_-\}$ which satisfy the (fairly) general set of initial conditions

$$y(x,0) \equiv y_0(x); \quad \dot{y}(x,0) \equiv v_0(x). \tag{14.3}$$

Note that the initial conditions for a continuous system are specified as *functions* rather than *numbers*. Substituting the solution (14.2) at $t = 0$,

$$\begin{aligned} y_0(x) &= y_+(x) + y_-(x) \\ v_0(x) &= -cy'_+(x) + cy'_-(x), \end{aligned} \tag{14.4}$$

where the $'$ indicates differentiation of y_+ and y_- with respect to their (different) arguments. Differentiating the first line in Eq. (14.4) with respect to x, dividing the second line by $-c$, and adding,

$$\begin{aligned} 2y'_+(x) &= y'_0(x) - \tfrac{1}{c}v_0(x) \\ y_+(x) &= \tfrac{1}{2}y_0(x) - \tfrac{1}{2c}\int_0^x du\, v_0(u) + C_+. \end{aligned}$$

Similarly,

$$y_-(x) = \tfrac{1}{2}y_0(x) + \tfrac{1}{2c}\int_0^x du\, v_0(u) + C_-,$$

where $C_+ + C_- = 0$. The full solution is then

$$\begin{aligned} y(x,t) = {} & \frac{y_0(x-ct) + y_0(x+ct)}{2} + \\ & + \frac{1}{2c}\int_{x-ct}^{x+ct} du\, v_0(u). \end{aligned} \tag{14.5}$$

14.2. Travelling sinusoidal waves.

A special case of the general solution (14.2) occurs when y_+ and y_- are harmonic functions. Allowing the amplitude, angular frequency, and phase of either wave to be arbitrary,

$$\begin{aligned} y_+(x-ct) &\equiv \Re\big(\tilde{A}_+ e^{ik_+(ct-x)}\big) \\ y_-(x+ct) &\equiv \Re\big(\tilde{A}_- e^{ik_-(ct+x)}\big). \end{aligned} \tag{14.6}$$

The constants $\tilde{A}_\pm$ are the *complex wave amplitudes*. The real constants $k_\pm$ have been introduced to make the exponents dimensionless.

Obviously the *time* dependence of Eq. (14.6) is of the form $\exp(i\omega_\pm t)$ with

$$\omega_\pm = ck_\pm. \tag{14.7}$$

The *spatial* dependence is of the similar form $\exp(\mp ik_\pm x)$. In a single (x) dimension, the $k_\pm$ are *wave numbers* $2\pi/\lambda_\pm$, where $\lambda_\pm$ are the wavelengths. In three dimensions, $k_\pm x$ is replaced by $\mathbf{k}_\pm \cdot \mathbf{x}$; the $\mathbf{k}_\pm$ are called *wave vectors*. The relation (14.7) between ω and k is called a *dispersion relation* (because the waves disperse if the relationship between ω and k is nonlinear).

As usual, it is customary and convenient to work directly with the complex displacement $\tilde{A}_\pm \exp\big(ik_\pm(ct \mp x)\big)$, rather than with its real part $y_\pm$.

The phenomenon of *beats* between sinusoidal waves is most simply demonstrated by the case of two waves of equal amplitude and phase but different wave number, travelling in the same ($+x$) direction:

$$y(x,t) = \Re\big(\tilde{A}e^{i(\omega_1 t - k_1 x)} + \tilde{A}e^{i(\omega_2 t - k_2 x)}\big). \tag{14.8}$$

Assuming $\Delta k \ll k_0$, define

$$\begin{aligned} k_0 &\equiv \tfrac{1}{2}(k_1 + k_2) \\ \Delta k &\equiv k_2 - k_1 \\ \omega_1 &\equiv \omega(k_1) \approx \omega_0 - \frac{\Delta k}{2} \frac{d\omega}{dk}\Big|_{k_0} \\ \omega_2 &\equiv \omega(k_2) \approx \omega_0 + \frac{\Delta k}{2} \frac{d\omega}{dk}\Big|_{k_0}. \end{aligned} \tag{14.9}$$

Here we have allowed for the possibility that $\frac{d\omega}{dk}$ is not always equal to a constant value c, as it is for a simple string. This occurs particularly on stiff strings (as on the upper octaves of a piano). Finally, define the *group velocity*

$$v_{\rm gr} \equiv \frac{d\omega}{dk}, \tag{14.10}$$

here not necessarily equal to the *phase velocity*

$$c \equiv \frac{\omega_0}{k_0}. \tag{14.11}$$

With these definitions, Eq. (14.8) becomes

$$\begin{aligned} y(x,t) &= \Re\Big\{\tilde{A}\Big(e^{i(\omega_0 t - k_0 x)}\big(e^{i\frac{\Delta k}{2}(v_{\rm gr}t - x)} + \\ &+ e^{-i\frac{\Delta k}{2}(v_{\rm gr}t - x)}\big)\Big)\Big\} \\ &= \Re\Big\{2\tilde{A}\Big(e^{i(\omega_0 t - k_0 x)} \cos\big(\tfrac{\Delta k}{2}(v_{\rm gr}t - x)\big)\Big)\Big\}. \end{aligned} \tag{14.12}$$

In this result, $\exp\big(i(\omega_0 t - k_0 x)\big)$ is the high-frequency short-wavelength *carrier wave* moving with speed c; $\cos\big(\frac{\Delta k}{2}(v_{\rm gr}t - x)\big)$ is the low-frequency long-wavelength *amplitude modulating wave* moving with speed $v_{\rm gr}$. Because it multiplies the carrier, this modulating wave is an *envelope* that determines the amplitude of the short carrier waves inside it. Since the information content is provided by these amplitude variations, the information is borne by the *modulation*: "the music travels with the group velocity".

14.3. Standing waves and normal modes.

Starting with a pair of sinusoidal waves as in Eq. (14.6), next we investigate the implications of requiring that the string be *fixed* at its two endpoints $x = 0$ and $x = L$. At the first endpoint,

$$\begin{aligned} 0 &= y(x{=}0, t) \\ &= \Re\big(\tilde{A}_+ e^{i\omega_+ t} + \tilde{A}_- e^{i\omega_- t}\big) \\ \Rightarrow \tilde{A}_+ &= -\tilde{A}_- \equiv \frac{\tilde{A}}{2i} \\ \Rightarrow \omega_+ &= \omega_- \equiv \omega; \quad k_+ = k_- \equiv k. \end{aligned}$$

With these substitutions, the displacement y is a *standing wave*

$$y(x,t) = \Re\big(\tilde{A} e^{i\omega t}\big) \sin kx.$$

At the second endpoint $x = L$,

$$\begin{aligned} 0 &= y(x{=}L, t) \\ &= \Re\big(\tilde{A} e^{i\omega t}\big) \sin kL \\ \Rightarrow 0 &= \sin kL \\ \Rightarrow k_n &= \frac{n\pi}{L}, \;\; n = 1, 2, 3\ldots \end{aligned}$$

The general solution is a sum of standing waves:

$$y(x,t) = \sum_{n=1}^{\infty} \Re\big(\tilde{A}_n e^{i\omega_n t}\big) \sin \frac{n\pi x}{L}, \tag{14.13}$$

where $\omega_n \equiv \omega(k_n)$ in general, and $\omega_n = n\pi c/L$ for a simple string satisfying the wave equation (14.1).

As is apparent from Eq. (14.13), standing waves are the *product* of a sinusoidal t dependence and a sinusoidal x dependence. Periodically, when $\tilde{A}e^{i\omega t}$ is pure imaginary, y vanishes at all positions x. Likewise, at equally spaced *nodes* where $kx = n\pi$, y vanishes at all times t. Although there is no hint of propagation to the left or right, each standing wave actually is the sum of right- and left-propagating travelling waves having the same frequency and amplitude. All that is required to force any wave to be a standing wave is that the string be fixed at some point.

Equation (14.13) strongly resembles the coupled oscillator Eq. (12.21), which expressed the motion of a generalized coordinate as a sum over normal mode oscillations. In fact, each of the standing waves on a string is a normal mode. There are an infinite number of normal modes because the string has an infinite number of degrees of freedom (it is equivalent to n equally spaced point masses connected by $(n+1)$ massless springs, as $n \to \infty$).

Aside from their infinite number, the normal modes of the simple string described by the wave equation (14.1) have the same properties as those of a coupled oscillator. Each is excited to a degree required by the initial conditions. Thereafter, each mode continues to oscillate without interference from any other mode. The energy given initially to that mode (on average divided equally between T and U) is retained within that mode forever.

14.4. Fourier expansion in normal modes.

We can use the initial conditions from Eq. (14.3) to evaluate the constants $\tilde{A}_n$ in Eq. (14.13):

$$\begin{aligned} y_0(x) &= \sum_n \Re(\tilde{A}_n) \sin k_n x \\ v_0(x) &= -\sum_n \omega_n \Im(\tilde{A}_n) \sin k_n x. \end{aligned} \tag{14.14}$$

Again we take advantage of *Fourier's trick*, already used in our solution (3.13) to the simple oscillator with a nonlinear driving force. For each line in Eq. (14.14), multiply both sides by $\frac{2}{L} \sin \frac{m\pi x}{L}$, $1 \le m \le \infty$, and integrate over x from 0 to L. Using the identity

$$\frac{2}{L} \int_0^L dx \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} = \delta_{mn}, \tag{14.15}$$

all but one term in each right-hand sum vanishes, leaving

$$\begin{aligned} \Re(\tilde{A}_m) &= \frac{2}{L} \int_0^L dx \sin \frac{m\pi x}{L} y_0(x) \\ -\omega_m \Im(\tilde{A}_m) &= \frac{2}{L} \int_0^L dx \sin \frac{m\pi x}{L} v_0(x). \end{aligned} \tag{14.16}$$

14.5. Nodes and antinodes.

For the string we have been considering, *nodes* (zeroes of y) occur by definition at the fixed points $x = 0$ and $x = L$. If the string is vibrating in a single normal mode with $n > 1$, there are additional nodes at intermediate points. For example, when $n = 3$, nodes are found at $x = \frac{L}{3}$ and $\frac{2L}{3}$ as well.

For the same string, *antinodes* (zeroes of $\frac{\partial y}{\partial x}$) are found at $x = \frac{L}{6}$, $\frac{L}{2}$, and $\frac{5L}{6}$. More generally, what physical condition would produce an antinode rather than a node at the boundary $x = L$? An example would be a massive string extending from $x = 0$ to $x = L$, connected to a *massless* string with the same tension extending from $x = L$ to $x = \infty$. Here the slope of the massive string must vanish at $x = L$: otherwise a finite transverse force would be exerted on the massless string, resulting in infinite acceleration.

Since massless strings are hard to find, antinodal boundary conditions are more frequently encountered in the *longitudinal* oscillations of elastic media – for example, sound waves in air. There the phase velocity[2] is $c^2 = E/\rho$, where ρ is the mass density and E, the *elastic modulus*, is the ratio of *stress* to *strain*. (We shall have more to say later about the definition of these quantities.) For a perfect gas, c^2 is equal to the inverse of the *adiabatic compressibility*.

The classic example of an antinodal boundary condition is found in an organ pipe – the closed end is a node, the open end is (approximately) an antinode. Here the fundamental normal mode has only one quarter of its wavelength contained within the pipe, as opposed to one half wavelength for a pipe closed at both ends. The ($\omega/2\pi = 30$ Hz) introductory rumble in *Also Sprach Zarathustra* can be sustained in an open organ pipe only 2.8 m long.

14.6. Reflections of waves at boundaries.

When the boundary conditions are simple (nodal or antinodal), reflections of transverse waves on a string can be determined easily by means of the "virtual string" construction. Consider first a nodal boundary ($y \equiv 0$ at $x = L$). Recall that the wave equation is solved by a smooth shape propagating e.g. along $+\hat{x}$. Consider such a shape incident from the left ($x < L$) upon the nodal boundary.

The construction consists of hypothesizing the existence of a "virtual string" in the forbidden (stringless) region $x > L$, and imagining that the same but *inverted* shape is incident upon the nodal boundary from the right. The two shapes are timed to coincide when they reach $x = L$; the resulting cancellation satisfies the nodal boundary condition $y = 0$ there. Since the boundary condition is satisfied, we have found the correct solution. As the erect shape disappears into the virtual region, the inverted shape leaves the virtual region and appears in the physical region. There the observed waveform is the sum of (what remains of) the incident shape plus (the part of) the inverted shape which has propagated in from the virtual region. After the incident shape has disappeared completely, the reflected shape is fully inverted.

Similarly, the boundary condition $\frac{\partial y}{\partial x} = 0$ at an *antinodal* boundary is satisfied by a *non-inverted* shape on the virtual string; the shape reflected from an antinodal boundary is *erect*.

When the boundary conditions are *not* simple, the reflected and transmitted amplitudes are determined by matching the waveforms on either side of the boundary. For example, when the boundary consists of an interface between two strings of equal tension τ but different mass per unit length μ, we identify two matching conditions. First, the string's *displacement* must be the same on either side, in order for the string to remain continuous. Second, the string's *slope* must also be the same on either side of the boundary. Otherwise there would be a finite net transverse force on the infinitesimal element of string at the boundary, resulting in an infinite acceleration.

Without loss of generality, we place the origin $x = 0$ at the interface between the two strings. The matching conditions are

$$\begin{aligned} y(x{=}0^-, t) &= y(x{=}0^+, t) \\ \frac{\partial y}{\partial x}(x{=}0^-, t) &= \frac{\partial y}{\partial x}(x{=}0^+, t). \end{aligned} \tag{14.17}$$

Denote the string displacement in the region $x < 0$ by y_1 and that in the region $x > 0$ by y_2. For $x < 0$ we must allow for both an incident wave $f_1(t - x/c_1)$ and a reflected wave $g_1(t + x/c_1)$, while for $x > 0$ we need only a transmitted wave $f_2(t - x/c_2)$. Equations (14.17) become

$$\begin{aligned} f_1(t) + g_1(t) &= f_2(t) \\ -\tfrac{1}{c_1} f_1'(t) + \tfrac{1}{c_1} g_1'(t) &= -\tfrac{1}{c_2} f_2'(t). \end{aligned} \tag{14.18}$$

Now differentiate both sides of the first equality with respect to t, multiply the second equality by c_2, and add:

$$\begin{aligned} f_1'(t) + g_1'(t) &= f_2'(t) \\ \tfrac{c_2}{c_1}\big(-f_1'(t) + g_1'(t)\big) &= -f_2'(t) \\ (c_2 + c_1) g_1'(t) &= (c_2 - c_1) f_1'(t) \\ g_1(t) &= \mathcal{R} f_1(t) + \text{const} \\ g_1(t) &= \mathcal{R} f_1(t) \\ \mathcal{R} &\equiv \frac{c_2 - c_1}{c_2 + c_1}. \end{aligned} \tag{14.19}$$

In the next to last line we set the constant of integration equal to zero because it does not represent a wave. In the last line we introduced the *reflected amplitude ratio* $\mathcal{R}$.

Similarly, the transmitted wave is

$$\begin{aligned} f_2(t) &= \mathcal{T} f_1(t) \\ \mathcal{T} &\equiv \frac{2c_2}{c_1 + c_2}, \end{aligned} \tag{14.20}$$

where $\mathcal{T}$ is the *transmitted amplitude ratio.* In terms of the incident wave f_1, the full solution is

$$\begin{aligned} y_1(x,t) &= f_1(t - x/c_1) + \mathcal{R} f_1(t + x/c_1) \\ y_2(x,t) &= \mathcal{T} f_1(t - x/c_2). \end{aligned} \tag{14.21}$$

For the above case in which the string tensions τ_1 and τ_2 are equal, $c_1 \propto \mu_1^{-1/2}$ and $c_2 \propto \mu_2^{-1/2}$ with the same constant of proportionality. Defining

$$Z_1 \equiv (\mu_1 \tau_1)^{-1/2} \quad Z_2 \equiv (\mu_2 \tau_2)^{-1/2}, \tag{14.22}$$

for this case $\tau_1 = \tau_2$ it is also possible to write

$$\mathcal{R} = \frac{Z_2 - Z_1}{Z_2 + Z_1} \quad \mathcal{T} = \frac{2Z_2}{Z_2 + Z_1}. \tag{14.23}$$

Had we worked the most general problem in which the string masses per unit length and the string tensions *both* are allowed to be different on either side of the interface, we would have obtained the results (14.21) and (14.23). The quantity $Z \equiv (\mu\tau)^{-1/2}$ is called the *characteristic impedance* of the string. It is usually more illuminating to characterize a wave medium by its phase velocity c and impedance Z, rather than by the less fundamental properties μ and τ.

It is easy to identify three limiting cases of Eqs. (14.21) and (14.23). When the impedance Z_2 vanishes (right-hand string is a brick wall), $\mathcal{R} = -1$ and $\mathcal{T} = 0$. This is equivalent to the nodal boundary for which we used the virtual string construction. When Z_2 is infinite (right-hand string is a massless filament), $\mathcal{R} = +1$ and $\mathcal{T} = 2$. The transmitted amplitude is *twice* the incident amplitude. This is equivalent to the antinodal boundary. Finally, when the characteristic impedances are the same on either side of the interface, even if the tensions and masses per unit length are different there is *no reflection*.

The reflection formulae (14.21) and (14.23) carry over to waves in other media. For example, considering electromagnetic waves in uniform isotropic media, the electric field $\mathbf{E}$ plays the role of the string displacement; the characteristic impedance is the ratio of $|\mathbf{E}|$ to $|\mathbf{H}|$. In vacuum this ratio is equal to 377 ohms. For electromagnetic waves in a coaxial cable, Z is equal to the same ratio, which is equivalent to $\sqrt{L/C}$, where L and C are the inductance and capacitance per unit length. This ratio is 138 ohms $\times \ln(b/a)$, where a and b are the inner and outer cable radii. Of course, for either of these electromagnetic examples, in vacuum the phase velocity c is the speed of light. When an electromagnetic wave travels in a *refractive* medium, both the phase velocity and the characteristic impedance are reduced by the factor $1/n$, where n is the *index of refraction*.

As a final example, consider a one dimensional Schrödinger wave that is incident on a barrier V which is smaller than its energy E. Here the wave's phase velocity is *directly* proportional to its wave number $k \equiv \sqrt{2m(E-V)}/\hbar$, while the characteristic impedance is *inversely* proportional to k.

15. Mechanics of solids.

15.1. Stress, strain, and waves in a one-dimensional solid.

To set the stage for consideration of a real (three-dimensional) elastic solid, first we consider a hypothetical medium in which the molecules are able to move only in one direction. Let $u(x)$ be the difference between the actual and the equilibrium position x of a molecule. The local distortion $N \equiv \frac{\partial u}{\partial x}$ of the medium is called the *strain*.

Creating a strain requires exerting a force on the medium. Consider a plane of area ΔA with its normal along x. In this one-dimensional case, the force exerted across that plane by one element of the medium upon another is oriented also in the x direction. This force per unit area S is called the *stress*.

The *elastic modulus* E of the medium measures the size of the stress required to produce a strain: $S = EN$. Since the strain is dimensionless, both the stress and the elastic modulus have units of *pressure*. In MKS this is the $\mathrm{N/m^2}$, or *pascal*; one atmosphere is $\approx 10^5$ pascals. Steel, one of the stiffer structural materials, has an elastic modulus of $\approx 2 \times 10^{11}$ pascals.

Consider a volume element $\Delta A \Delta x$ of the medium, to which a stress S is applied to create a strain N. Suppose that the molecules at $x = 0$ are held stationary while the molecules at $x = \Delta x$, already displaced from their equilibrium positions by u, are further displaced by du. Before this extra displacement, the strain is $N = \frac{u}{\Delta x}$. The work done corresponding to du is

$$\begin{aligned} dW &= F\,du = S\Delta A\,du \\ &= EN\Delta A\,du = E\tfrac{u}{\Delta x}\Delta A\,du \\ W &= \tfrac{1}{2}E\tfrac{\Delta A}{\Delta x}u^2 = \tfrac{1}{2}E\,\Delta A\Delta x\,N^2. \end{aligned}$$

Per unit volume, the potential energy U' associated with the strain is $\frac{1}{2}EN^2 = \frac{1}{2}E(\frac{\partial u}{\partial x})^2$.

The kinetic energy T' per unit volume is $\frac{1}{2}\rho(\frac{\partial u}{\partial t})^2$, where ρ is the mass density. Therefore, for this solid in which the molecules move only along x, the Lagrangian per unit volume, or Lagrangian density, is

$$\mathcal{L}' = \tfrac{1}{2}\rho\big(\frac{\partial u}{\partial t}\big)^2 - \tfrac{1}{2}E\big(\frac{\partial u}{\partial x}\big)^2. \tag{15.1}$$

This result is fully analagous to the string's Lagrangian density (13.5). The same application of the Euler-Lagrange equation (13.2) yields a wave equation for this hypothetical solid:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} + \frac{\rho}{E}\frac{\partial^2 u}{\partial t^2} &= 0\\ \sqrt{\frac{E}{\rho}} &\equiv c,\end{aligned} \tag{15.2}$$

where c is the phase velocity. This is essentially the same as the wave equation for the string. However, in the solid, the molecular displacement is *along* the direction of wave propagation, as is true for a *longitudinal wave*, rather than *transverse* to that direction as for the string.

15.2. Stress and strain tensors in a solid.

All of our remaining discussion of solids parallels that of section 15.1, except that the solid is no longer hypothetical – its molecules will be allowed to move in three dimensions. This greatly complicates the mathematics, but the basic ideas remain the same.

In a solid or a flowing viscous liquid, forces $\Delta\mathbf{F}$ that act across a surface $\Delta\mathbf{A}$ can have components which are parallel as well as perpendicular to the surface. The stress $\mathcal{S}$ relating one to the other must be a *second rank tensor*:

$$\begin{aligned}\Delta\mathbf{F} &\equiv \mathcal{S}\cdot\Delta\mathbf{A}\\ (\Delta F)_i &= \mathcal{S}_{ij}\Delta A_j.\end{aligned} \tag{15.3}$$

Here the first stress index i refers to the Cartesian component of the force, while the second index j refers to the component of the *normal* to the area being considered. As usual, summation over repeated indices is implied.

The stress tensor must be *symmetric* for a static solid. This can be appreciated by considering the four x_1 or x_2 faces of a cube of material within the solid. If $\mathcal{S}_{21}$ were greater than $\mathcal{S}_{12}$, the cube would experience a net torque in the x_3 direction.

In three dimensions, the displacement $\mathbf{u}$ of each molecule from its equilibrium position is a *vector field* depending on x_1, x_2, and x_3. Here the strain must be defined to take into account all three components of $\mathbf{u}$ and of $\mathbf{x}$: it is also a second rank tensor. In analogy to the one-dimensional strain,

$$\mathcal{N}_{ij} \equiv \tfrac{1}{2}\big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\big). \tag{15.4}$$

In Eq. (15.4) we defined the strain tensor to be *manifestly symmetric*. A possible component of the form

$$\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}$$

cannot be allowed to contribute. If it were nonzero but small, such a component would correspond to an infinitesimal rotation. This could not be part of a strain, because it is not a deformation of the solid – no stress would be required to produce it.

15.3. Fourth-rank tensor of elasticity.

In the one-dimensional case, the elastic modulus was the constant of proportionality relating the (scalar) stress to the (scalar) strain. In three dimensions, both the stress and strain are symmetric tensors. Even for isotropic materials, it will turn out that the six independent elements of $\mathcal{S}$ and of $\mathcal{N}$ do not all have the same constant of proportionality to each other. Therefore the two must be related by a *fourth rank tensor, the elasticity* $\mathcal{E}$:

$$\begin{aligned}\mathcal{S} &\equiv \mathcal{E}\mathcal{N}\\ \mathcal{S}_{ij} &= \mathcal{E}_{ijkl}\mathcal{N}_{kl}.\end{aligned} \tag{15.5}$$

In analogy to the one-dimensional potential energy per unit volume,

$$U' = \tfrac{1}{2}NEN\ ,$$

the three-dimensional potential energy density is

$$\begin{aligned} U' &= \tfrac{1}{2}\mathcal{N}_{ij}\mathcal{E}_{ijkl}\mathcal{N}_{kl} \\ &= \tfrac{1}{8}\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\Big)\mathcal{E}_{ijkl}\Big(\frac{\partial u_k}{\partial x_l}+\frac{\partial u_l}{\partial x_k}\Big). \end{aligned} \tag{15.6}$$

The kinetic energy density is more straightforward:

$$\mathcal{T}' = \tfrac{1}{2}\rho\frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t}. \tag{15.7}$$

As for the one-dimensional case, the Lagrangian density $\mathcal{L}'$ is $T' - U'$.

15.4. Elasticity in a homogeneous isotropic solid.

A homogeneous isotropic solid, unlike any crystalline structure, has a fourth-rank tensor of elasticity with components that, by definition, are independent of particular axis directions. The only available building block, aside from scalars, is the unit matrix I with components δ_{ij}. The most general fourth rank tensor that we are able to construct is

$$\mathcal{E}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{15.8}$$

The last term is manifestly symmetric under interchange of i and j in order to ensure that $\mathcal{S}$ is symmetric. Because of the homogeneity and isotropy, the two values λ and μ, called the *Lamé constants*, are sufficient to determine all 81 elements of $\mathcal{E}$.

Evaluating the stress,

$$\begin{aligned} \mathcal{S}_{ij} &= \mathcal{E}_{ijkl}\mathcal{N}_{kl} \\ &= \lambda\delta_{ij}\delta_{kl}\mathcal{N}_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\mathcal{N}_{kl} \\ &= \lambda\delta_{ij}\,\mathrm{tr}\,\mathcal{N} + 2\mu\mathcal{N}_{ij}. \end{aligned} \tag{15.9}$$

For example, comparing the first and third equalities in Eq. (15.9), one has

$$\begin{aligned} &\mathcal{S}_{11} = \lambda(\mathcal{N}_{11} + \mathcal{N}_{22} + \mathcal{N}_{33}) + 2\mu\mathcal{N}_{11} \\ &\Rightarrow \mathcal{E}_{1111} = \lambda + 2\mu \\ &\Rightarrow \mathcal{E}_{1122} = \lambda \\ &\mathcal{S}_{12} = 2\mu\mathcal{N}_{12} = \mathcal{E}_{1212}\mathcal{N}_{12} + \mathcal{E}_{1221}\mathcal{N}_{21} \\ &\Rightarrow \mathcal{E}_{1212} = \mathcal{E}_{1221} = \mu. \end{aligned} \tag{15.10}$$

It is instructive to visualize the distortion that, for example, is controlled by the elasticity element $\mathcal{E}_{1111}$. Consider a solid cube of volume l^3 with faces normal to $\hat{x}_1$, $\hat{x}_2$, or $\hat{x}_3$. Suppose that the the cube is stretched in the $\hat{x}_1$ direction, so that the *diagonal* strain element $\mathcal{N}_{11}$ is positive. This stretching is caused by a stress $\mathcal{S}_{11}$. Further assume that *no* other distortion is present: *all* the other elements of $\mathcal{N}$ vanish. This means that *additional* stresses must be exerted in order to prevent the cube from shrinking in the $\hat{x}_2$ and $\hat{x}_3$ directions, as would be its natural tendency. In other words, forces must be exerted to hold the sides $x_2 =$ constant and $x_3 =$ constant at their equilibrium separation l. The fact that $\mathcal{E}_{1111} = \lambda + 2\mu$ (Eq. (15.10)) means that the effective elastic modulus Y_{eff} for this particular stretching mode is equal to $\lambda + 2\mu$.

Similarly, for the same cube, consider a *shear displacement* of the four faces normal to $\hat{x}_1$ or $\hat{x}_2$ so that the face normal to $\hat{x}_3$ changes in shape from a square to a rhombus. The four sides of the rhombus make angles $\pm\Delta\phi/2$ with the edges of the original square. This describes a distortion corresponding to nonzero *off-diagonal* strain elements $\mathcal{N}_{12} = \mathcal{N}_{21} = \Delta\phi/2$. All other possible distortions (and strain elements) are assumed to vanish. Because $\mathcal{E}_{1212} = 2\mu$, the stress $\mathcal{S}_{12} = \mathcal{S}_{21}$ which must be applied to create this shear displacement is

$$\begin{aligned} \mathcal{S}_{12} &= \mathcal{E}_{1212}\mathcal{N}_{12} + \mathcal{E}_{1221}\mathcal{N}_{21} \\ &= \mu\frac{\Delta\phi}{2} + \mu\frac{\Delta\phi}{2} = \mu\Delta\phi. \end{aligned}$$

15.5. Young's modulus and Poisson's ratio.

The strains and stresses described in the last two paragraphs may be straightforward to visualize, but they are not the easiest quantities to measure. A more practical approach to determining the Lamé constants is to experiment with a solid rectangular bar, for example having relaxed square cross section w^2 and length l. One such experiment measures *Young's modulus* Y, the ratio of the pressure F/w^2 with which the ends are pulled apart to the fractional increase $\Delta l/l$ in the bar's length. Another measures *Poisson's ratio* σ, the (negative) ratio of the fractional change in w to that in l for the same experiment:

$$\begin{aligned} Y &\equiv \frac{F/w^2}{\Delta l/l} \\ \sigma &\equiv -\frac{\Delta w/w}{\Delta l/l}. \end{aligned} \tag{15.11}$$

For both experiments, *no* constraint is imposed on the side walls of the bar.

The price paid for simplifying these measurements is the task of finding λ and μ, given Y and σ. Taking $\hat{x}_1$ to be the long axis of the bar, we know that the only stress applied is $\mathcal{S}_{11}$. Only the diagonal elements of the strain are nonzero: since the bar is not twisted, by symmetry the off-diagonal elements must vanish. By definition of the Poisson ratio σ, $\mathcal{N}_{22} = \mathcal{N}_{33} \equiv -\sigma\mathcal{N}_{11}$. And by definition of Young's modulus Y, $\mathcal{S}_{11} \equiv Y\mathcal{N}_{11}$. Starting from Eq. (15.9),

$$\begin{aligned} \mathcal{S}_{11} &= \lambda(\mathcal{N}_{11} + \mathcal{N}_{22} + \mathcal{N}_{33}) + 2\mu\mathcal{N}_{11} \\ Y &= \lambda(1 - \sigma - \sigma) + 2\mu \\ \mathcal{S}_{22} &= \lambda(\mathcal{N}_{11} + \mathcal{N}_{22} + \mathcal{N}_{33}) + 2\mu\mathcal{N}_{22} \\ 0 &= \lambda(1 - \sigma - \sigma) - 2\mu\sigma. \end{aligned}$$

Subtracting the fourth from the second equality,

$$\begin{aligned} Y &= 2\mu(1+\sigma) \\ \mu &= \frac{Y}{2(1+\sigma)}. \end{aligned} \tag{15.12}$$

With Eq. (15.12), the second equality alone yields

$$\begin{aligned} \lambda &= \frac{2\mu\sigma}{1-2\sigma} \\ &= \frac{\sigma Y}{(1+\sigma)(1-2\sigma)}. \end{aligned} \tag{15.13}$$

Equations (15.12) and (15.13) express the Lamé constants in terms of Young's modulus and Poisson's ratio. The inverse equations are

$$\begin{aligned} \sigma &= \frac{\lambda}{2(\lambda+\mu)} \\ Y &= \mu\frac{3\lambda+2\mu}{\lambda+\mu}. \end{aligned} \tag{15.14}$$

Note that $0 \le \sigma \le \frac{1}{2}$. The upper limit is obtained either if $\lambda = \infty$ (medium is incompressible) or if $\mu = 0$ (medium cannot support a shear stress). Mechanical engineers often *assume* that $\sigma = \frac{1}{2}$ when better data are not available. If the medium is incompressible, Young's modulus is *not* infinite; shear displacement still allows the bar to elongate, with modulus $Y = 3\mu$.

15.6. Waves in solids.

So far we have been discussing the *statics* of three-dimensional solids. As an introduction to the *dynamics*, we consider wave propagation in a solid medium. As was the case for waves on a string, we neglect the damping forces that cause the wave eventually to die out. For a cast bell with $Q \approx 10^3$, able to ring for seconds at a frequency of hundreds of Hz, this is a good approximation.

The Lagrangian density is given by the difference of Eqs. (15.7) and (15.6). Since the elasticity tensor $\mathcal{E}_{ijkl}$ is symmetric under the interchanges $i \leftrightarrow j$ and $k \leftrightarrow l$ for a solid that is homogeneous and isotropic, we may replace $\mathcal{N}_{ij}$ and $\mathcal{N}_{kl}$ in Eq. (15.6) by $\partial u_i/\partial x_j$ and $\partial u_k/\partial x_l$. The potential energy density becomes

$$\begin{aligned} U' &= \tfrac{1}{2}\frac{\partial u_i}{\partial x_j}\big(\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\big)\frac{\partial u_k}{\partial x_l} \\ &= \frac{\lambda}{2}\frac{\partial u_i}{\partial x_i}\frac{\partial u_j}{\partial x_j} + \frac{\mu}{2}\big(\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\big). \end{aligned} \tag{15.15}$$

The Lagrangian density is independent of $\mathbf{u}$. Since its dependence on $\partial u/\partial t$ is confined to T', and its dependence upon $\partial u/\partial x$ is confined to U', the Euler-Lagrange equation in the n^{th} component of $\mathbf{u}$ becomes

$$\frac{d}{dt}\frac{\partial T}{\partial(\frac{\partial u_n}{\partial t})} = \frac{d}{dx_k}\frac{\partial U}{\partial(\frac{\partial u_n}{\partial x_k})}. \tag{15.16}$$

Using Eq. (15.7), the left hand side is

$$\frac{d}{dt}\big(\rho(\mathbf{x},t)\frac{\partial u_n}{\partial t}\big) = \rho\frac{\partial^2 u_n}{\partial t^2} + \frac{\partial\rho}{\partial t}\frac{\partial u_n}{\partial t}.$$

Since the time-varying component of ρ is assumed to be a small fraction of its average, the second term is negligible compared to the first.

The right hand side of Eq. (15.16) is

$$\begin{aligned}
&\lambda\frac{d}{dx_n}(\frac{\partial u_j}{\partial x_j})+\mu\frac{d}{dx_k}(\frac{\partial u_n}{\partial x_k}+\frac{\partial u_k}{\partial x_n})\\
=&\lambda\frac{\partial^2 u_j}{\partial x_n\partial x_j}+\mu(\frac{\partial^2 u_n}{\partial x_k\partial x_k}+\frac{\partial^2 u_k}{\partial x_k\partial x_n})\\
=&(\lambda+\mu)\frac{\partial}{\partial x_n}\frac{\partial u_k}{\partial x_k}+\mu\frac{\partial^2 u_n}{\partial x_k\partial x_k}\\
=&(\lambda+\mu)\frac{\partial}{\partial x_n}(\nabla\cdot\mathbf{u})+\mu\nabla^2 u_n.
\end{aligned}$$

Considered together, the Euler-Lagrange equations for $n = 1$, 2, and 3 are equivalent to the vector equation

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2}=(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})+\mu\nabla^2\mathbf{u}. \qquad (15.17)$$

Suppose first that the displacement field $\mathbf{u}$ is *divergenceless*, so that there can be no compression. Then any wave still present is a *shear wave*. Equation (15.17) becomes

$$\mu\nabla^2\mathbf{u}-\rho\frac{\partial^2\mathbf{u}}{\partial t^2}=0, \qquad (15.18)$$

satisfied by a shear wave with phase velocity $\sqrt{\mu/\rho}$.

Conversely, suppose that $\mathbf{u}$ is *curlless* so that there can be no shear. Using the "*bac cab*" rule,

$$\begin{aligned}
0&=\nabla\times(\nabla\times\mathbf{u})\\
&=\nabla(\nabla\cdot\mathbf{u})-\nabla^2\mathbf{u},
\end{aligned}$$

Eq. (15.17) becomes

$$(\lambda+2\mu)\nabla^2\mathbf{u}-\rho\frac{\partial^2\mathbf{u}}{\partial t^2}=0. \qquad (15.19)$$

This describes a *compression wave* with the larger phase velocity $\sqrt{(\lambda+2\mu)/\rho}$. An earthquake including both compression and shear waves might be felt first as a sharp compressive jolt, followed by a rolling shear motion.

16. Mechanics of fluids.

16.1. Static fluids.

A fluid (liquid or gas) differs from a solid in that it cannot support a shear stress if it is static (i.e. if the fluid is not moving). In terms of the Lamé constants, $\mu \equiv 0$, so that Young's modulus $Y = 0$ and Poisson's ratio $\sigma = \frac{1}{2}$.

Consider an infinitesimal cube of side l with one corner at the origin. The $\hat{x}_1$ component of the force $\mathbf{F}$ exerted on the cube is

$$\begin{aligned}
F_1=l^2\{&\big(\mathcal{S}_{11}(l,x_2,x_3)-\mathcal{S}_{11}(0,x_2,x_3)\big)+\\
&+\big(\mathcal{S}_{12}(x_1,l,x_3)-\mathcal{S}_{12}(x_1,0,x_3)\big)+\\
&+\big(\mathcal{S}_{13}(x_1,x_2,l)-\mathcal{S}_{13}(x_1,x_2,0)\big)\}\\
&=l^3\frac{\partial}{\partial x_j}\mathcal{S}_{1j}.
\end{aligned}$$

Defining $\mathbf{f}$ to be the force per unit volume,

$$\begin{aligned}
f_i&=\frac{\partial}{\partial x_j}\mathcal{S}_{ij}\\
\tilde{f}^t&=\tilde{\partial}^t\mathcal{S},
\end{aligned} \qquad (16.1)$$

where $\tilde{\partial}^t$ is a row vector with elements equal to $\partial/\partial x_j$, and, as usual, a sum is taken over the repeated index j.

For a static fluid, the stress tensor has no off-diagonal (shear) components. Also, the fluid's homogeneity requires the diagonal elements to be equal. Then

$$\mathcal{S}\equiv -p\,\mathrm{I},$$

where p is the *pressure* and I is the unit matrix. Equation (16.1) reduces to

$$\mathbf{f}=-\nabla p.$$

We add a (conservative) external force $-\rho\nabla\phi$ derived from ϕ, a *potential per unit mass*. Since ρ is the mass per unit volume, this additional term is also a force per unit volume. Insisting that the total force on any static element of fluid vanish, we obtain the basic equation of fluid statics

$$0=\mathbf{f}=-\nabla p-\rho\nabla\phi. \qquad (16.2)$$

The two most common applications of Eq. (16.2) involve an external force due to gravity, for which the gravitational potential is $\phi = gz$, where z is the vertical coordinate. For an *incompressible* fluid (ρ = constant), integrating Eq. (16.2),

$$\begin{aligned} 0 &= -\nabla p - \rho\nabla(gz) \\ -p_0 &= -p - \rho g(z - z_0) \\ p &= p_0 - \rho g(z - z_0). \end{aligned}$$

This leads directly to *Archimedes' Principle*, which states that the buoyant force on a (partially or totally) submerged object is the weight of the water that it displaces.

For a *perfect gas* at constant temperature, $\rho = \rho_0 p/p_0$. Equation (16.2) yields

$$\begin{aligned} 0 &= -\nabla p - \frac{\rho_0 p}{p_0}\nabla(gz) \\ \frac{\partial p/\partial z}{p} &= -\frac{\rho_0 g}{p_0} \\ p &= p_0 e^{-\rho_0 g(z-z_0)/p_0}. \end{aligned}$$

Applying this result to the earth's atmosphere, where $\rho_0 \approx 1$ kg/m^3, $p_0 \approx 10^5$ pascals, and $g \approx 10$ m/sec^2, we estimate that the atmosphere's pressure is reduced by a factor e after an elevation gain of $p_0/\rho_0 g \approx 10^4$ m.

16.2. Flow of a nonviscous fluid.

Fluids that are not static *do* support a shear stress. Temporarily we choose to neglect this fact, focusing on "dry" or runny liquids. Now that it accelerates the fluid, we no longer require the force per unit volume in Eq. (16.2) to vanish:

$$\begin{aligned} -\nabla p - \rho\nabla\phi &= \rho\frac{d\mathbf{v}}{dt} \\ -\frac{\nabla p}{\rho} - \nabla\phi &= \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}. \end{aligned} \qquad (16.3)$$

This is the *Navier-Stokes equation*. On the right hand side of the last equality we have written the total time derivative of $\mathbf{v}$ as a *convective derivative*. The first term describes the explicit change of velocity with time ("Spring approaches and the river flows ever faster"). The second term describes the change in $\mathbf{v}$ due to flow of the fluid ("water moves from a pool to the rapids").

We are faced with two unknown scalar fields (p and ρ) and one unknown vector field ($\mathbf{v}$), but we have derived only one vector equation (16.3). We need two additional scalar equations. The first is an *equation of state* relating ρ to p, as in the two examples of section (16.1). The second is the *equation of continuity* expressing the conservation of fluid molecules:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0. \qquad (16.4)$$

16.3. *Steady* flow of an *incompressible* nonviscous fluid.

Under the assumption of *steady flow*, the fluid can be moving, but its velocity field, and every other characteristic of the fluid, possesses no *explicit* time dependence. That is, $\partial/\partial t$ of *anything* vanishes. Also in this section we assume that the fluid is incompressible (ρ = constant). The Navier-Stokes equation (16.3) becomes

$$-\nabla\Big(\frac{p}{\rho} + \phi\Big) = (\mathbf{v}\cdot\nabla)\mathbf{v}.$$

Using the vector identity

$$\tfrac{1}{2}\nabla(\mathbf{v}\cdot\mathbf{v}) = (\mathbf{v}\cdot\nabla)\mathbf{v} + \mathbf{v}\times(\nabla\times\mathbf{v}),$$

the above equation may be rewritten as

$$\nabla\Big(\frac{p}{\rho} + \phi + \frac{v^2}{2}\Big) = \mathbf{v}\times(\nabla\times\mathbf{v}). \qquad (16.5)$$

Taking the dot product of $\mathbf{v}$ with Eq. (16.5),

$$\begin{aligned} 0 &= (\mathbf{v}\cdot\nabla)\Big(\frac{p}{\rho} + \phi + \frac{v^2}{2}\Big) \\ &= \Big(\frac{d}{dt} - \frac{\partial}{\partial t}\Big)\Big(\frac{p}{\rho} + \phi + \frac{v^2}{2}\Big) \\ &= \frac{d}{dt}\Big(\frac{p}{\rho} + \phi + \frac{v^2}{2}\Big). \end{aligned} \qquad (16.6)$$

This is the first form of *Bernoulli's equation*. It says that the quantity $p/\rho + \phi + v^2/2$ for an element of fluid is constant as the fluid moves – it is *constant along streamlines*. Bernoulli's equation is an expression of *energy conservation* for the fluid, with p/ρ and ϕ the *potential* terms, and $\frac{1}{2}v^2$ the *kinetic*. It is responsible for the lift on airplane wings and the operation of spray nozzles.

16.3. Steady *irrotational* flow of an incompressible nonviscous fluid.

In this section we make the still stronger assumption that the flow is *irrotational*, $\nabla \times \mathbf{v} \equiv 0$. Then the right-hand side of Eq. (16.5) vanishes altogether, and

$$\frac{p}{\rho} + \phi + \frac{v^2}{2} = \text{constant} \tag{16.7}$$

throughout the entire fluid. This is the second form of Bernoulli's equation.

With this strong set of assumptions, we can solve for $\mathbf{v}$ without even considering the Navier-Stokes equation. For steady flow the first term in the continuity equation (16.4) vanishes, and for an incompressible fluid the second term reduces to

$$\nabla \cdot \mathbf{v} = 0. \tag{16.8}$$

If the flow is irrotational, we may write $\mathbf{v} = -\nabla\xi$, where ξ is the *velocity potential*. Equation (16.8) is equivalent to *Laplace's equation*

$$\nabla^2\xi = 0. \tag{16.9}$$

Solving Laplace's equation is a standard problem in applied mathematics. A unique solution exists whenever ξ or its normal derivative is specified over an entire closed surface, part of which may be at infinity. For fluid flow problems, it is more common to specify the normal derivative of ξ (velocity normal to the boundary) than it is to specify ξ itself. Analytic solutions to Laplace's equation may be obtained by *series expansions* involving harmonic and hyperbolic functions in Cartesian coordinates, Bessel and Neumann functions in cylindrical coordinates, or spherical harmonics in spherical coordinates. For solutions of Laplace's equations in two dimensions, *conformal transformations* are useful. Numerical solutions to Laplace's equation can be obtained by a variety of methods. A simple procedure is to set up a square or cubic grid and demand (by iterative solution) that ξ at each grid point be equal to the average of its nearest neighbors.

16.5. Flow of a viscous fluid.

At last we no longer neglect the shear stress in a flowing fluid. Consider two parallel planes of fluid with area A separated by Δz along their normal. Suppose that the top plane is moving with velocity $\Delta\mathbf{v} = \Delta v_y \hat{y}$ while the bottom plane is stationary. The *first coefficient of viscosity* η is defined by

$$\frac{F}{A} \equiv \eta \frac{\Delta v_y}{\Delta z}, \tag{16.10}$$

where F is the force in the y direction exerted by the top plane on the bottom plane. Identifying F/A with a stress $\mathcal{S}_{yz}$, for this simple example Eq. (16.10) may be written

$$\mathcal{S}_{yz} = \eta \frac{\partial v_y}{\partial z}.$$

Symmetrizing,

$$\mathcal{S}_{ij} \equiv \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{16.11}$$

This is the formal definition of η for an *incompressible* fluid, in which viscous forces are caused only by shear flow.

Considering once more the strain $\mathcal{N}$ defined by Eq. (15.4), it is clear that Eq. (16.11) may alternatively be written

$$\mathcal{S}_{ij} = 2\eta \frac{\partial \mathcal{N}_{ij}}{\partial t}. \tag{16.12}$$

What happens if the fluid is both viscous and compressible? Resistance to rapid compression will be exerted as a result of its viscosity, in addition to that from its Lamé constant λ. Equation (16.12) must be extended:

$$\mathcal{S}_{ij} = 2\eta \frac{\partial \mathcal{N}_{ij}}{\partial t} + \eta' \delta_{ij} \frac{\partial}{\partial t} \mathrm{tr}\, \mathcal{N}, \tag{16.13}$$

where η' is the *second coefficient of viscosity*. Notice the similarity between Eq. (15.9) for a homogeneous isotropic solid and Eq. (16.13) for a compressible liquid: apart from redefinition of constants, the right hand side of (16.13) is just the partial time derivative of that in (15.9). Note also that the coefficient of $\eta'\delta_{ij}$ in (16.13) is just $\nabla \cdot \mathbf{v}$, which vanishes for an incompressible fluid according to Eq. (16.8).

The above are the mathematical definitions. The physical picture is that thermal motion of molecules causes them to be exchanged between the strata of relatively moving fluid. These exchanged molecules retain their original average velocity in the direction of slippage, tending to equalize the velocities of the strata. One analogy is that of two open coal trains passing each other with workers shoveling coal back and forth.

In a perfect gas at fixed temperature, η is independent of pressure p. This is because the mean free path is inversely proportional to p, while the density is directly proportional to p; the viscosity is proportional to their product.

16.6. Vorticity.

The force $\mathbf{f}$ per unit volume on a viscous fluid is the usual term $-\nabla p - \rho\nabla\phi$ from Eq. (16.2) plus a term $\mathbf{f}^{\mathrm{v}}$ from the viscosity. Combining Eqs. (16.1) and (16.13), and neglecting possible spatial variation of the viscosity,

$$\begin{aligned} f_i^{\mathrm{v}} &= \frac{\partial}{\partial x_j}\big(\eta(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) + \eta'\delta_{ij}(\nabla\cdot\mathbf{v})\big) \\ &= \eta\frac{\partial}{\partial x_j}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) + \eta'\frac{\partial}{\partial x_i}(\nabla\cdot\mathbf{v}) \\ &= \eta\nabla^2 v_i + (\eta+\eta')\frac{\partial}{\partial x_i}(\nabla\cdot\mathbf{v}) \\ \mathbf{f}^{\mathrm{v}} &= \eta\nabla^2\mathbf{v} + (\eta+\eta')\nabla(\nabla\cdot\mathbf{v}). \end{aligned} \tag{16.14}$$

Adding $\mathbf{f}^{\mathrm{v}}$ to the Navier-Stokes equation,

$$\begin{aligned} &-\frac{\nabla p}{\rho} - \nabla\phi + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{\eta+\eta'}{\rho}\nabla(\nabla\cdot\mathbf{v}) \ = \\ &= \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}. \end{aligned} \tag{16.15}$$

Exploiting the same vector identity used to prove Eq. (16.5), the last term in Eq. (16.15) may be rewritten as $\frac{1}{2}\nabla v^2 - \mathbf{v}\times(\nabla\times\mathbf{v})$. Our plan is take the *curl* of Eq. (16.15) after this modification. All the terms proportional to gradients will vanish. Again assuming that the fluid is incompressible so that ρ is constant, we are left with the curl of

$$\frac{\eta}{\rho}\nabla^2\mathbf{v} = \frac{\partial\mathbf{v}}{\partial t} - \mathbf{v}\times(\nabla\times\mathbf{v}). \tag{16.16}$$

Taking said curl, and defining the *vorticity* $\vec{\Omega} \equiv \nabla\times\mathbf{v}$, we obtain

$$\begin{aligned} \frac{\eta}{\rho}\nabla^2\vec{\Omega} &= \frac{\partial\vec{\Omega}}{\partial t} - \nabla\times(\mathbf{v}\times\vec{\Omega}) \\ &= \frac{\partial\vec{\Omega}}{\partial t} + \nabla\times(\vec{\Omega}\times\mathbf{v}). \end{aligned} \tag{16.17}$$

In a nonviscous fluid for which $\eta = 0$, the left hand side vanishes. It will be seen that the right hand side can lead to *persistent vortices* in the fluid.

16.7. Diffusion of the vorticity: Reynolds number.

Invoking a similar vector identity,

$$\begin{aligned} \nabla\times(\vec{\Omega}\times\mathbf{v}) = (\mathbf{v}\cdot\nabla)\vec{\Omega} - (\vec{\Omega}\cdot\nabla)\mathbf{v} \,+ \\ + \vec{\Omega}(\nabla\cdot\mathbf{v}) - \mathbf{v}(\nabla\cdot\vec{\Omega}), \end{aligned}$$

we see that the last term in the identity vanishes by definition of $\vec{\Omega}$, and the second last term vanishes according to the continuity equation (16.4) when the fluid is incompressible. Equation (16.17) becomes

$$\begin{aligned} \frac{\eta}{\rho}\nabla^2\vec{\Omega} &= \frac{\partial\vec{\Omega}}{\partial t} + (\mathbf{v}\cdot\nabla)\vec{\Omega} - (\vec{\Omega}\cdot\nabla)\mathbf{v} \\ &= \frac{d\vec{\Omega}}{dt} - (\vec{\Omega}\cdot\nabla)\mathbf{v} \\ \frac{d\vec{\Omega}}{dt} &= \frac{\eta}{\rho}\nabla^2\vec{\Omega} + (\vec{\Omega}\cdot\nabla)\mathbf{v}. \end{aligned} \tag{16.18}$$

In some cases the velocity has no spatial dependence along the direction of the vorticity. For example, $\mathbf{v}$ may be oriented in the x-y plane and may depend only upon x and y, in which case $\vec{\Omega}$ is oriented along $\hat{z}$. In these special instances, Eq. (16.18) reduces to

$$\frac{d\vec{\Omega}}{dt} = \frac{\eta}{\rho}\nabla^2\vec{\Omega}. \tag{16.19}$$

This is a *diffusion equation* for each component of the vorticity, in which the diffusion constant $D \equiv \eta/\rho$. Unlike more traditional applications of this equation, e.g. the diffusion of impurity gas molecules, Eq. (16.19) describes the diffusion of *vorticity*. The small whirlpool formed when a rower's oar is lifted from the water of a still pond diffuses outward into a larger, slower vortex and eventually disappears. On the other hand, Eq. (16.19) requires the vorticity to *persist* if η/ρ may be neglected.

Since η/ρ has units of $\mathrm{m}^2/\mathrm{sec}$, we must construct another quantity of the same dimensions with which to compare it. Consider a pipe of circular cross section with diameter d, in which fluid flows with average velocity V. Then Vd has the same dimensions as η/ρ. The ratio of the two is called the *Reynolds number* R_e:

$$R_e \equiv \frac{\rho V d}{\eta}. \tag{16.20}$$

The Reynolds number is large when η/ρ is small, and vice versa.

With the help of Eq. (16.19), we can guess what happens in the pipe. If R_e is small, the velocity $\eta/\rho d$ characterizing diffusion of the vorticity is much faster than the flow velocity. If a small vortex is present, it spreads out quickly along the pipe and dissipates. The absence of vortices results in *laminar flow*, for which the velocity field is easy to calculate. In smooth pipes, laminar flow occurs for Reynolds numbers up to surprisingly large values, of order a few hundred. But when R_e is large (above 10,000), an element of fluid flows so fast that the vorticity cannot spread out rapidly enough to escape it. This is *turbulent flow*. It is so difficult to model that research in this area is a frontier of mathematical physics. For example, it is virtually impossible to calculate and predict the eddies that cause a flag to flutter.

– End –

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

ASSIGNMENT 1

Reading:
105 Notes 1.1, 1.2, 1.3, 1.4, 1.5.
Hand & Finch 7.1, 7.2, 7.3, 7.4, and 8.7 (pp. 300-302 only).

1. A matrix A is called *orthogonal* if

$$A^{-1} = A^t \ ,$$

where

$$\left(A^t\right)_{ij} \equiv A_{ji} \ .$$

(a) Prove that the product of two orthogonal matrices is also orthogonal.
(b) Show that if A is a 3×3 orthogonal matrix, its three column vectors are mutually perpendicular and of unit length.

2. Suppose that a vector $\mathbf{x}'$ in the space axes is related to a vector $\mathbf{x}$ in the body axes by

$$\mathbf{x}' = A\mathbf{x} \ ,$$

where A is a transformation matrix. Given a matrix F, find a matrix F', expressed in terms of F and A, such that

$$\mathbf{x}'^t F' \mathbf{x}' = \mathbf{x}^t F \mathbf{x} \ .$$

F and F' are said to be related by a *similarity transformation*.

3. Define the *trace* of a matrix F as

$$\mathrm{Tr}(F) = F_{ij}\delta_{ij} \ ,$$

where, as usual, summation over repeated indices is implied.
(a) Show that $\mathrm{Tr}(F)$ is the sum of the diagonal elements of F.
(b) Prove that $\mathrm{Tr}(F)$ is invariant under any similarity transformation.

4.
(a) Use the Levi-Civita density ϵ_{ijk} to prove the *bac cab rule*

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \ .$$

(b) Use the bac cab rule to show that

$$\mathbf{a} = \hat{\mathbf{n}}(\mathbf{a} \cdot \hat{\mathbf{n}}) + \hat{\mathbf{n}} \times (\mathbf{a} \times \hat{\mathbf{n}}) \ ,$$

where $\hat{\mathbf{n}}$ is any unit vector. What is the geometrical significance of each of the two terms in the expansion?

5. Consider three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.
(a) Show that

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \epsilon_{ijk} u_i v_j w_k \ ,$$

where, as usual, summation is implied.
(b) If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ emanate from a common point, show that $|\epsilon_{ijk} u_i v_j w_k|$ is the volume of the parallelepiped whose edges they determine.

6. In a complex vector space, a matrix U is called *unitary* if

$$U^{-1} = U^\dagger \ ,$$

where

$$\left(U^\dagger\right)_{ij} \equiv U^*_{ji} \ .$$

Show than an *infinitesimal* unitary transformation T (one that is infinitesimally different from the unit matrix) can be written

$$T \approx I + iH \ ,$$

where I is the unit matrix and H is *Hermitian*, *i.e.*

$$H = H^\dagger \ .$$

7. Show that $\mathbf{v}$, $\mathbf{p}$, and $\mathbf{E}$ (velocity, momentum, and electric field) are ordinary ("polar") vectors, while $\boldsymbol{\omega}$, $\mathbf{L}$, and $\mathbf{B}$ (angular velocity, angular momentum, and magnetic field) are pseudo ("axial") vectors.

8. Find the transformation matrix Λ, such that

$$x'_i = \Lambda_{ij} x_j \ ,$$

which describes the following (passive) transformation: relative to the space (primed) axes, the body (unprimed) axes are rotated counterclockwise by an angle ξ about a unit vector $\hat{\mathbf{n}}'$ which has direction cosines n'_1, n'_2, and 0 with respect to the x'_1, x'_2, and x'_3 (space) axes, respectively.

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO PROBLEM SET 1
Solutions by T. Bunn and M. Strovink

Reading:
105 Notes 1.1, 1.2, 1.3, 1.4, 1.5.
Hand & Finch 7.1, 7.2, 7.3, 7.4, and 8.7 (pp. 300-302 only).

1. A matrix A is called *orthogonal* if

$$A^{-1} = A^t \ ,$$

where

$$\left(A^t\right)_{ij} \equiv A_{ji} \ .$$

(a) Prove that the product of two orthogonal matrices is also orthogonal.
Solution: Note: We're going to use the summation convention all the time in these solution sets, so you'd better get used to it. Just remember, repeated indices get summed over, unless specifically indicated otherwise. For instance, when we say $(AB)_{ij} = A_{ik}B_{kj}$, the right-hand side means $\sum_{k=1}^{n} A_{ik}B_{kj}$. It's really not so hard.

Also, we've explained the reasoning used in problem 1(a) in more detail than we usually will, for those of you who aren't used to matrix manipulations. Make sure you understand each step of this reasoning.

Suppose A and B are orthogonal, so their transposes are the same as their inverses. Then we need to show that $(AB)^t = (AB)^{-1}$. In other words, we want to show that $(AB)(AB)^t$ is the identity matrix. Well, here goes: The i,j element of this product is

$$\begin{aligned}
((AB)(AB)^t)_{ij} &\overset{\mathbf{1}}{=} (AB)_{ik}(AB)^t_{kj} \\
&\overset{\mathbf{2}}{=} (AB)_{ik}(AB)_{jk} \\
&\overset{\mathbf{3}}{=} A_{il}B_{lk}A_{jm}B_{mk} \\
&\overset{\mathbf{4}}{=} A_{il}A_{jm}B_{lk}B^t_{km} \\
&\overset{\mathbf{5}}{=} A_{il}A_{jm}(BB^t)_{lm} \\
&\overset{\mathbf{6}}{=} A_{il}A_{jm}\delta_{lm} \\
&\overset{\mathbf{7}}{=} A_{il}A_{jl} \\
&\overset{\mathbf{8}}{=} A_{il}A^t_{lj} \\
&\overset{\mathbf{9}}{=} \delta_{ij} \ .
\end{aligned}$$

But δ_{ij} is the i,j element of the identity matrix, so we've shown what we intended to.

Now, as promised, here's an explanation of each "=" sign in the above: (1) This is the ordinary rule for matrix multiplication. (2) The "t" is for "transpose": It means switch the order of the two indices k and j. (3) Matrix multiplication rule applied to each of the two terms in parentheses. (4) Transpose. (5) Matrix multiplication rule (applied in reverse this time). (6) B is orthogonal, so BB^t is the identity, and the l,m element of the identity is δ_{lm}. (7) Substitution rule for the δ symbol. (8) Transpose. (9) A is orthogonal.

(You don't need to write out all of these steps in so much detail. Also, if you know an easier way to do it, that's fine. In particular, if you know that the transpose of AB is B^tA^t, there's a really easy way to do part (a): $(AB)(AB)^t = ABB^tA^t = ABB^{-1}A^{-1} = AA^{-1} = I$.)

(b) Show that if A is a 3×3 orthogonal matrix,

its three column vectors are mutually perpendicular and of unit length.
Solution: Define $\mathbf{V}^j$ to be the j^{th} column vector of A, *i.e.* $(V^j)_i = A_{ij}$. Then

$$\begin{aligned}\mathbf{V}^j \cdot \mathbf{V}^k &= (V^j)_i (V^k)_i \\ &= A_{ij} A_{ik} \\ &= (A^t)_{ki} A_{ij} \\ &= (A^t A)_{kj} \\ &= I_{kj} \\ &= \delta_{kj} \ ,\end{aligned}$$

where I is the identity matrix.

2. Suppose that a vector $\mathbf{x}'$ in the space axes is related to a vector $\mathbf{x}$ in the body axes by

$$\mathbf{x}' = A\mathbf{x} \ ,$$

where A is a transformation matrix. Given a matrix F, find a matrix F', expressed in terms of F and A, such that

$$\mathbf{x}'^t F' \mathbf{x}' = \mathbf{x}^t F \mathbf{x} \ .$$

F and F' are said to be related by a *similarity transformation.*
Solution: If $\mathbf{x}' = A\mathbf{x}$, then $\mathbf{x} = A^{-1}\mathbf{x}'$. (To see why, multiply both sides of the first equation by A^{-1} on the left.) Using the fact that the transpose of AB is $B^t A^t$, we get $\mathbf{x}^t = \mathbf{x}'^t (A^{-1})^t = \mathbf{x}'^t A$. (For the last step, remember that transformation matrices like A are orthogonal.) Now, for any matrix F, $\mathbf{x}^t F \mathbf{x} = \mathbf{x}'^t A F A^{-1} \mathbf{x}'$. (We just substituted for $\mathbf{x}$ and $\mathbf{x}^t$.) So we can choose the matrix F' to be AFA^{-1}.

3. Define the *trace* of a matrix F as

$$\text{Tr}(F) = F_{ij}\delta_{ij} \ ,$$

where, as usual, summation over repeated indices is implied.
(a) Show that $\text{Tr}(F)$ is the sum of the diagonal elements of F.
Solution: Remember the substitution rule for δ_{ij}: If you have an expression containing a δ_{ij}, and the i is being summed over, then you can simply replace it by j and get rid of the δ_{ij}. In this case, the expression $F_{ij}\delta_{ij}$ is the same as F_{jj}. The index j is repeated, so it's being summed over. So F_{jj} means "the sum of all the elements of F that have the same row index and column index." That sounds the same as "sum of all the diagonal elements".

(b) Prove that $\text{Tr}(F)$ is invariant under any similarity transformation.
Solution: We're being asked to show that if F and F' are related by a similarity transformation, then $\text{Tr}(F) = \text{Tr}(F')$. Well, if F and F' are similar, then we can write $F' = AFA^{-1}$ for some A. That means that the i,j element of F' is $F'_{ij} = A_{ik} F_{kl} A^{-1}_{lj}$. So by the trace rule:

$$\begin{aligned}\text{Tr}(F') &= A_{ik} F_{kl} A^{-1}_{lj} \delta_{ij} \\ &= A_{ik} F_{kl} A^{-1}_{li} \\ &= (A^{-1}A)_{lk} F_{kl} \\ &= \delta_{lk} F_{kl} = \text{Tr}(F) \ .\end{aligned}$$

4.
(a) Use the Levi-Civita density ϵ_{ijk} to prove the *bac cab rule*

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \ .$$

Solution:

$$\begin{aligned}(\mathbf{b} \times \mathbf{c})_k &\equiv \epsilon_{klm} b_l c_m \\ (\mathbf{a} \times (\mathbf{b} \times \mathbf{c}))_i &\equiv \epsilon_{ijk} a_j (\mathbf{b} \times \mathbf{c})_k \\ &= \epsilon_{ijk} a_j (\epsilon_{klm} b_l c_m) \\ &= \epsilon_{ijk} \epsilon_{klm} a_j b_l c_m \ .\end{aligned}$$

Any nonvanishing term must have i, j, l, and m all different from k, but every index must take one of only three values. The possibilities are:

- $i = l$, $j = m$: the nonvanishing term $(k \neq i, k \neq j)$ has value $+1$, because $\epsilon_{ijk} = \epsilon_{klm}$ are cyclic permutations of each other.
- $i = m$, $j = l$: the nonvanishing term $(k \neq i, k \neq j)$ has value -1, because $\epsilon_{ijk} = \epsilon_{klm}$ are *not* cyclic permutations of each other.

Changing these words into an equation in the second line below,

$$\begin{aligned}(\mathbf{a}\times(\mathbf{b}\times\mathbf{c}))_i &= \epsilon_{ijk}\epsilon_{klm}a_j b_l c_m \\ &= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})a_j b_l c_m \\ &= a_j b_i c_j - a_j c_i b_j \\ &= b_i(a_j c_j) - c_i(a_j b_j) \\ &= b_i(\mathbf{a}\cdot\mathbf{c}) - c_i(\mathbf{a}\cdot\mathbf{b})\ .\end{aligned}$$

This is bac cab rule for the i^{th} component. Nothing is special about this component, so the rule is proved in general.

(b) Use the bac cab rule to show that

$$\mathbf{a} = \hat{\mathbf{n}}(\mathbf{a}\cdot\hat{\mathbf{n}}) + \hat{\mathbf{n}}\times(\mathbf{a}\times\hat{\mathbf{n}})\ ,$$

where $\hat{\mathbf{n}}$ is any unit vector. What is the geometrical significance of each of the two terms in the expansion?
Solution: by the bac cab rule,

$$\begin{aligned}\hat{\mathbf{n}}\times(\mathbf{a}\times\hat{\mathbf{n}}) &= \mathbf{a}(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}) - \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{a}) \\ &= \mathbf{a} - \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{a}) \\ \mathbf{a} &= \hat{\mathbf{n}}(\mathbf{a}\cdot\hat{\mathbf{n}}) + \hat{\mathbf{n}}\times(\mathbf{a}\times\hat{\mathbf{n}})\ .\end{aligned}$$

This expression splits the vector $\mathbf{a}$ into two parts, the first of which is parallel to $\hat{\mathbf{n}}$, and the second of which is perpendicular to it.

5. Consider three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.
(a) Show that

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \epsilon_{ijk}u_i v_j w_k\ ,$$

where, as usual, summation is implied.
Solution: In component language, the dot product rule is $\mathbf{u}\cdot\mathbf{v} = u_i v_i$, and the cross product rule is $(\mathbf{u}\times\mathbf{v})_k = \epsilon_{ijk}u_i v_j$. (The only way to see that this rule for cross products is right is to check it explicitly. Let's see that it works for the z-component: $(\mathbf{u}\times\mathbf{v})_3 = \epsilon_{123}u_1v_2 + \epsilon_{213}u_2v_1 = u_1v_2 - u_2v_1$, which is right.)

So $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = (\mathbf{u}\times\mathbf{v})_k w_k = \epsilon_{ijk}u_i v_j w_k$.

(b) If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ emanate from a common point, show that $|\epsilon_{ijk}u_i v_j w_k|$ is the volume of the parallelepiped whose edges they determine.
Solution: Using the result of (a),

$$\begin{aligned}|\epsilon_{ijk}u_i v_j w_k| &= |\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}| \\ &= |(\mathbf{u}\times\mathbf{v})||\mathbf{w}||\cos\theta_w| \\ &= |\mathbf{u}||\mathbf{v}||\sin\theta_{uv}|\mathbf{w}||\cos\theta_w|\ ,\end{aligned}$$

where θ_{uv} is the angle between $\mathbf{u}$ and $\mathbf{v}$, and θ_w is the angle between $\mathbf{w}$ and the normal to the plane defined by $\mathbf{u}$ and $\mathbf{v}$. The first three factors yield the area of the parallelogram defined by $\mathbf{u}$ and $\mathbf{v}$, and the last two factors yield the height of the parallelepiped whose base is this parallelogram. The product of the base area and the height is the volume of the parallelepiped.

6. In a complex vector space, a matrix U is called *unitary* if

$$U^{-1} = U^\dagger\ ,$$

where

$$\left(U^\dagger\right)_{ij} \equiv U^*_{ji}\ .$$

Show than an *infinitesimal* unitary transformation T (one that is infinitesimally different from the unit matrix) can be written

$$T \approx I + iH\ ,$$

where I is the unit matrix and H is *Hermitian*, *i.e.*

$$H = H^\dagger\ .$$

Solution:

$$\begin{aligned}T &= I + iH \\ T^\dagger &= I^\dagger + (iH)^\dagger \\ &= I + H^\dagger i^\dagger \\ &= I + H^\dagger(-i) \\ &= I - iH^\dagger\ ,\end{aligned}$$

where in the second line we used the fact that, as for the transpose, $(AB)^\dagger = B^\dagger A^\dagger$. Enforcing the unitarity condition on T,

$$\begin{aligned} T^{-1} &= T^\dagger \\ I &= T^{-1}T \\ &= T^\dagger T \\ &= (I - iH^\dagger)(I + iH) \\ &= I + i(H - H^\dagger) + \mathcal{O}(H^2) \\ &\approx I + i(H - H^\dagger) \\ 0 &\approx i(H - H^\dagger) \\ H^\dagger &\approx H \ . \end{aligned}$$

7. Show that $\mathbf{v}$, $\mathbf{p}$, and $\mathbf{E}$ (velocity, momentum, and electric field) are ordinary ("polar") vectors, while $\boldsymbol{\omega}$, $\mathbf{L}$, and $\mathbf{B}$ (angular velocity, angular momentum, and magnetic field) are pseudo ("axial") vectors.
Solution: Obviously $\mathbf{r} = (x_1, x_2, x_3)$, the position vector, changes sign under parity inversion $\mathcal{P}$, which transforms $(x_1 \to -x_1, x_2 \to -x_2, x_3 \to -x_3)$. Therefore $\mathbf{r}$ is a polar vector. Since $\mathbf{v} \equiv d\mathbf{r}/dt$ and $\mathbf{p} = m\mathbf{v}$, $\mathbf{v}$ and $\mathbf{p}$ are also polar vectors. So is the force $\mathbf{F} = d\mathbf{p}/dt$. The force on a test charge q is $\mathbf{F} = q\mathbf{E}$, so $\mathbf{E}$ must also be a polar vector.

However (see 105 lecture notes eq. (1.7)), $v'_{\text{tang}} = \boldsymbol{\omega} \times \mathbf{r}$ implies that $\boldsymbol{\omega}$ cannot change sign under $\mathcal{P}$, inasmuch as v'_{tang} and $\mathbf{r}$ both do change sign. Therefore $\boldsymbol{\omega}$ is an axial vector. Similarly, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is an axial vector because both $\mathbf{r}$ and $\mathbf{p}$ change sign under $\mathcal{P}$. Likewise, the magnetic part of the force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ requires $\mathbf{B}$ to be an axial vector because $\mathbf{F}$ and $\mathbf{v}$ both change sign under $\mathcal{P}$.

8. Find the transformation matrix Λ, such that

$$x'_i = \Lambda_{ij} x_j \ ,$$

which describes the following (passive) transformation: relative to the space (primed) axes, the body (unprimed) axes are rotated counterclockwise by an angle ξ about a unit vector $\hat{\mathbf{n}}'$ which has direction cosines n'_1, n'_2, and 0 with respect to the x'_1, x'_2, and x'_3 (space) axes, respectively.
Solution: It is somewhat easier to find Λ^t such that $x_i = \Lambda^t_{ij} x'_j$, as is the case for the Euler rotation (105 lecture notes 1.5). We accomplish the transformation from the primed to the unprimed coordinates in three steps:

(i) Rotate the x'_1 and x'_2 axes about the x'_3 axis so that the new "1" direction (call it x''_1) is along the given unit vector $\hat{\mathbf{n}}'$. Call the rotation matrix which accomplishes this A.

(ii) Rotate the x''_2 and x''_3 axes counterclockwise about the x''_1 (or $\hat{\mathbf{n}}'$) axis by the given angle ξ. Call the matrix which accomplishes this B.

(iii) Reverse the rotation (i). This is necessary so that, for example, if ξ is zero there will have been no net change. Call the matrix which accomplishes this C.

We will then have $\Lambda^t = CBA$, or $\Lambda = A^t B^t C^t$. What are the elements of these rotation matrices? Remembering that we are dealing with *passive* rotations (the axes get rotated, not the vectors), rotation A is

$$A = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \ ,$$

where ϕ is the angle whose cosine is n'_1 and whose sine is n'_2. (For getting A right, the only tricky part is to figure out where the minus sign goes. It is straightforward to do this by considering the effect of operating A on $\hat{\mathbf{n}}'$ itself: the result should be, and is, entirely in the "1" direction.) Similarly, rotation matrix B is

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\xi & \sin\xi \\ 0 & -\sin\xi & \cos\xi \end{pmatrix} \ .$$

C is merely the inverse of A: $C = A^t$. Then $\Lambda = A^t B^t A$. Substituting $\cos\phi = n'_1$, $\sin\phi = n'_2$ and multiplying the matrices,

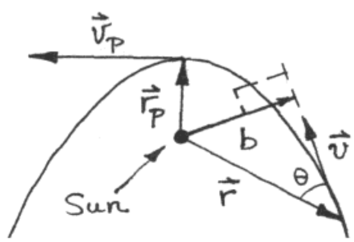
$$\Lambda = \begin{pmatrix} (n'_1)^2 + (n'_2)^2 \cos\xi & n'_1 n'_2 (1 - \cos\xi) & n'_2 \sin\xi \\ n'_1 n'_2 (1 - \cos\xi) & (n'_2)^2 + (n'_1)^2 \cos\xi & -n'_1 \sin\xi \\ -n'_2 \sin\xi & n'_1 \sin\xi & \cos\xi \end{pmatrix} \ .$$

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

ASSIGNMENT 2

Reading:
105 Notes 2.1, 2.2, 2.3, 2.4, 2.5.
Hand & Finch pp. 10-12, 130-134, 284-285.

1.
A comet, barely unbound by the sun (its total energy vanishes), executes a parabolic orbit about it.



At a certain time the comet has speed v and impact parameter b with respect to the sun. You may neglect the comet's mass m with respect to the sun's mass M. Find the perigee (distance of closest approach to the sun) of the comet.

2.
Two masses m_1 and m_2 orbit around their common center of mass (CM), which has the coordinate $\mathbf{R}(t)$. They are separated from the CM by $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$, respectively. Define

$$\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$$
$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} ,$$

where μ is the *reduced mass.*
(**a**)
Show that the total kinetic energy $T = T_1 + T_2$ is equal to

$$T = \tfrac{1}{2}\mu\dot{\mathbf{r}}^2 + \tfrac{1}{2}M\dot{\mathbf{R}}^2 ,$$

where $M = m_1 + m_2$.
(**b**)
About the CM, show that the total angular momentum $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$ is equal to

$$\mathbf{L} = \mu\mathbf{r} \times \dot{\mathbf{r}} .$$

The simplicity of these formulæ explains why the two-particle separation $\mathbf{r}$ and the two-particle reduced mass μ are usually chosen as parameters for analysis of the two-body problem.

3.
Two particles connected by an elastic string of stiffness k and equilibrium length b rotate about their center of mass with angular momentum l. Show that their distances of closest and furthest approach, r_1 and r_2, are related by

$$r_1^2 r_2^2 (r_1 + r_2 - 2b) = (r_1 + r_2) l^2/k\mu ,$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the two-body reduced mass.

4.
Determine which of the following forces are conservative, and find the potential energy (within a constant) for those which are:
(**a**)

$$F_x = 6abyz^3 - 20bx^3y^2$$
$$F_y = 6abxz^3 - 10bx^4y$$
$$F_z = 18abxyz^2 .$$

(**b**)

$$F_x = 18abyz^3 - 20bx^3y^2$$
$$F_y = 18abxz^3 - 10bx^4y$$
$$F_z = 6abxyz^2 .$$

(**c**)

$$\mathbf{F} = \hat{\mathbf{x}}F_1(x) + \hat{\mathbf{y}}F_2(y) + \hat{\mathbf{z}}F_3(z) .$$

5.
A vector field $\mathbf{F}$ is expressed in cylindrical coordinates as follows:

$$F_r = 0$$
$$F_\varphi = k/r$$
$$F_z = 0 ,$$

where k is a constant.

(**a**)

When $r > 0$, show that $\mathbf{F}$ has zero curl.

(**b**)

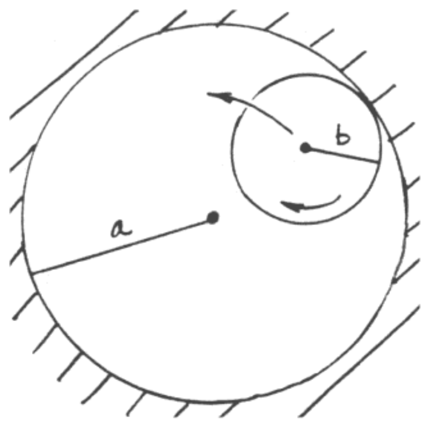
Consider the loop integral $\oint \mathbf{F} \cdot d\mathbf{l}$ counterclockwise around a circular path of fixed radius R. What is the value of this integral?

(**c**)

Can $\mathbf{F}$ be derived from a single-valued potential U, *e.g.* $\mathbf{F} = -\nabla U$ where $U = U(r, \varphi, z)$? Why or why not?

6.

A hoop of radius b and mass m rolls without slipping within a circular hole of radius $a > b$. About the center of the hole, the point of contact has a uniform angular velocity ω_a.



(**a**)

Find the angular velocity ω_b of the hoop about *its own* center (magnitude and direction).

(**b**)

Calculate the kinetic energy T of the hoop.

(**c**)

Obtain the angular momentum L (magnitude and direction) of the hoop relative to the center of the hole.

(**d**)

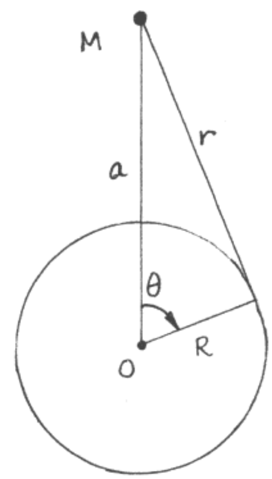
Consider your answers to (**b**) and (**c**) in the limit $b \to a$. You should find that both T and L vanish. Is this reasonable? Why or why not?

7.

Consider a spherically symmetric distribution $\rho(r)$ of mass density. If the gravitational acceleration, or "gravitational field vector" $\mathbf{g}$, is known to be independent of the radial coordinate r within a spherical volume, find $\rho(r)$ to within a multiplicative constant.

8.

Consider a point mass that lies outside a spherical surface. Let $\phi(\mathbf{r})$ be the gravitational potential due to the point mass.



Show that the average value of ϕ taken over the spherical surface is the same as the value of ϕ at the center of the sphere. [Since the potential due to an arbitrary mass distribution is the sum of potentials due to point masses, this statement is also true for the gravitational potential due to an arbitrary mass distribution lying outside a spherical surface.]

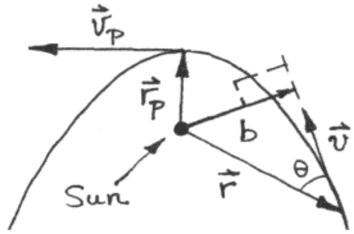
University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO PROBLEM SET 2

Solutions by T. Bunn and M. Strovink

Reading:
105 Notes 2.1, 2.2, 2.3, 2.4, 2.5.
Hand & Finch pp. 10-12, 130-134, 284-285.

1.
A comet, barely unbound by the sun (its total energy vanishes), executes a parabolic orbit about it.



At a certain time the comet has speed v and impact parameter b with respect to the sun. You may neglect the comet's mass m with respect to the sun's mass M. Find the perigee (distance of closest approach to the sun) of the comet.
Solution:
At the initial time, the angular momentum is

$$l = |\mathbf{r} \times \mathbf{p}| = mvr|\sin\theta| = mvb \ ,$$

where θ is the angle between $\mathbf{r}$ and $\mathbf{v}$, and $b \equiv r|\sin\theta|$. At the perigee, $l = mv_p r_p$, where v_p and r_p are the velocity and radius there. So, by angular momentum conservation, $v_p = vb/r_p$.

What about energy? As the problem states, a "barely bound" orbit is one with zero total energy, *i.e.* with kinetic energy exactly balancing potential energy. So we can set the energy at the perigee equal to zero:

$$\tfrac{1}{2}mv_p^2 - \frac{GMm}{r_p} = 0 \ .$$

Substitute for v_p:

$$\frac{v^2b^2}{2r_p^2} = \frac{GM}{r_p} \ .$$

Solving for r_p,

$$r_p = \frac{v^2b^2}{2GM} \ .$$

2.
Two masses m_1 and m_2 orbit around their common center of mass (CM), which has the coordinate $\mathbf{R}(t)$. They are separated from the CM by $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$, respectively. Define

$$\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$$
$$\mu \equiv \frac{m_1m_2}{m_1+m_2} \ ,$$

where μ is the *reduced mass.*
(a)
Show that the total kinetic energy $T = T_1 + T_2$ is equal to

$$T = \tfrac{1}{2}\mu\dot{\mathbf{r}}^2 + \tfrac{1}{2}M\dot{\mathbf{R}}^2 \ ,$$

where $M = m_1 + m_2$.
Solution:
First let's get what we can out of the definition of the CM:

$$\begin{aligned} \mathbf{r} &\equiv \mathbf{r}_1 - \mathbf{r}_2 \\ 0 &\equiv m_1\mathbf{r}_1 + m_2\mathbf{r}_2 \\ \Rightarrow \mathbf{r}_1 &= +\mathbf{r}\frac{m_2}{m_1+m_2} \\ \mathbf{r}_2 &= -\mathbf{r}\frac{m_1}{m_1+m_2} \ . \end{aligned}$$

Using these results to evaluate the kinetic energy,

$$
\begin{aligned}
T &= T_{\text{of CM}} + T_{\text{wrt CM}} \\
T_{\text{of CM}} &= \tfrac{1}{2} M \dot{\mathbf{R}}^2 \\
T_{\text{wrt CM}} &= \tfrac{1}{2} m_1 \dot{\mathbf{r}}_1^2 + \tfrac{1}{2} m_2 \dot{\mathbf{r}}_2^2 \\
&= \frac{\dot{\mathbf{r}}^2}{2(m_1+m_2)^2}(m_1 m_2^2 + m_2 m_1^2) \\
&= \frac{\dot{\mathbf{r}}^2}{2(m_1+m_2)} m_1 m_2 \\
&\equiv \tfrac{1}{2}\mu \dot{\mathbf{r}}^2 \\
T &= \tfrac{1}{2} M \dot{\mathbf{R}}^2 + \tfrac{1}{2}\mu \dot{\mathbf{r}}^2 \ .
\end{aligned}
$$

(**b**)
About the CM, show that the total angular momentum $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$ is equal to

$$\mathbf{L} = \mu \mathbf{r} \times \dot{\mathbf{r}} \ .$$

Solution:
Here we are concerned only with the angular momentum about the CM.

$$
\begin{aligned}
\mathbf{L} &= \mathbf{L}_1 + \mathbf{L}_2 \\
\mathbf{L}_1 &= m_1 \mathbf{r}_1 \times \dot{\mathbf{r}}_1 \\
\mathbf{L}_2 &= m_2 \mathbf{r}_2 \times \dot{\mathbf{r}}_2 \ .
\end{aligned}
$$

Using the results from part (**a**) for $\mathbf{r}_1$ and $\mathbf{r}_2$,

$$
\begin{aligned}
\mathbf{L} &= \frac{m_1 m_2^2}{(m_1+m_2)^2}\mathbf{r}\times\dot{\mathbf{r}} + \frac{m_2 m_1^2}{(m_1+m_2)^2}\mathbf{r}\times\dot{\mathbf{r}} \\
&= \frac{m_1 m_2 (m_1+m_2)}{(m_1+m_2)^2}\mathbf{r}\times\dot{\mathbf{r}} \\
&= \frac{m_1 m_2}{(m_1+m_2)}\mathbf{r}\times\dot{\mathbf{r}} \\
&\equiv \mu \mathbf{r}\times\dot{\mathbf{r}} \ .
\end{aligned}
$$

The simplicity of these formulæ explains why the two-particle separation $\mathbf{r}$ and the two-particle reduced mass μ are usually chosen as parameters for analysis of the two-body problem.

3.
Two particles connected by an elastic string of stiffness k and equilibrium length b rotate about their center of mass with angular momentum l. Show that their distances of closest and furthest approach, r_1 and r_2, are related by

$$r_1^2 r_2^2 (r_1 + r_2 - 2b) = (r_1 + r_2) l^2 / k\mu \ ,$$

where $\mu = m_1 m_2/(m_1+m_2)$ is the two-body reduced mass.
Solution:
Remember that a problem with two bodies and a central force can always be treated as a one-body problem: Just replace the mass by the reduced mass μ, and use the separation $\mathbf{r}$ between the bodies as your coordinate. Generically, if r is the distance from the origin and ω is the angular velocity, then the angular momentum l and energy E are

$$
\begin{aligned}
l &= \mu\omega r^2 \\
E &= \tfrac{1}{2}\mu\dot{r}^2 + \tfrac{1}{2}\mu(r\omega)^2 + \tfrac{1}{2}k(r-b)^2 \ ,
\end{aligned}
$$

where the last term is the potential energy in the stretched string. Let's use the equation for l to eliminate ω:

$$E = \frac{1}{2}\left(\mu\dot{r}^2 + \frac{l^2}{\mu r^2} + k(r-b)^2\right) \ .$$

When r is an extremum (at r_1 and r_2), $\dot{r} = 0$, so, setting the energies at r_1 and r_2 equal, and doing some algebra, we get

$$
\begin{aligned}
\frac{l^2}{\mu r_1^2} + k(r_1-b)^2 &= \frac{l^2}{\mu r_2^2} + k(r_2-b)^2 \\
\frac{l^2(r_2^2 - r_1^2)}{\mu r_1^2 r_2^2} &= k\left((r_2-b)^2 - (r_1-b)^2\right) \\
r_1^2 r_2^2 (r_1 + r_2 - 2b) &= \frac{l^2}{k\mu}(r_1+r_2) \ .
\end{aligned}
$$

4.
Determine which of the following forces are conservative, and find the potential energy (within a constant) for those which are:
(**a**)

$$
\begin{aligned}
F_x &= 6abyz^3 - 20bx^3y^2 \\
F_y &= 6abxz^3 - 10bx^4y \\
F_z &= 18abxyz^2 \ .
\end{aligned}
$$

Solution:
Remember that $F_x = -\partial U/\partial x$, and similarly

for F_y and F_z. So (if appropriate constants of integration are chosen) $U = -\int F_x\,dx = -\int F_y\,dy = -\int F_z\,dz$. If those three integrals aren't equal, then $\mathbf{F}$ can't be derived from a potential, and is not conservative. An equivalent statement is that $\mathbf{F}$ is conservative if and only if all the "mixed derivatives" are equal (*i.e.*, if $\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$, $\frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}$, and $\frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}$). "Equal mixed derivatives" is the same as "zero curl".

For part (**a**), if you work out all the mixed derivatives, you find that they are all OK (so the force *is* conservative). However, since we have to find U anyway, we'll just do the integrals:

$$\begin{aligned} U &= -\int F_x\,dx \\ &= -(6abxyz^3 - 5bx^4y^2 + A(y,z)) \\ &= -\int F_y\,dy \\ &= -(6abxyz^3 - 5bx^4y^2 + B(x,z)) \\ &= -\int F_z\,dz \\ &= -(6abxyz^3 + C(x,y)) \ , \end{aligned}$$

where A, B, and C are integration constants. (Note that they only have to be constant with respect to the variable of integration.) We can see that if we set $A = B = 0$, and $C = -5bx^4y^2$, we've got it. So

$$U = -6abxyz^3 + 5bx^4y^2 \ .$$

(**b**)

$$\begin{aligned} F_x &= 18abyz^3 - 20bx^3y^2 \\ F_y &= 18abxz^3 - 10bx^4y \\ F_z &= 6abxyz^2 \ . \end{aligned}$$

Solution:
This force is *not* conservative. It can't be, because $\frac{\partial F_x}{\partial z} \neq \frac{\partial F_z}{\partial x}$:

$$\begin{aligned} \frac{\partial F_x}{\partial z} &= 54abyz^2 \ , \text{ but} \\ \frac{\partial F_z}{\partial x} &= 6abyz^2 \ . \end{aligned}$$

(**c**)

$$\mathbf{F} = \hat{\mathbf{x}}F_1(x) + \hat{\mathbf{y}}F_2(y) + \hat{\mathbf{z}}F_3(z) \ .$$

Solution:
This force is conservative, since all mixed derivatives are equal (in fact, they're all zero). The potential is

$$U = -\int F_1(x)\,dx - \int F_2(y)\,dy - \int F_3(z)\,dz \ .$$

5.
A vector field $\mathbf{F}$ is expressed in cylindrical coordinates as follows:

$$\begin{aligned} F_r &= 0 \\ F_\varphi &= k/r \\ F_z &= 0 \ , \end{aligned}$$

where k is a constant.
(**a**)
When $r > 0$, show that $\mathbf{F}$ has zero curl.
Solution:
Transforming $\mathbf{F}$ to Cartesian coordinates,

$$\begin{aligned} F_z &= 0 \\ F_x &= F_r\cos\varphi - F_\varphi\sin\varphi \\ &= 0 - \tfrac{k}{r}\sin\varphi \\ &= \frac{kr\sin\varphi}{r^2} \\ &= -\frac{ky}{x^2+y^2} \ . \end{aligned}$$

Similarly

$$F_y = +\frac{kx}{x^2+y^2} \ .$$

Obviously $(\nabla\times\mathbf{F})_x = (\nabla\times\mathbf{F})_y = 0$ because $\mathbf{F}$ has no z component and no z dependence. The surviving component is

$$\begin{aligned} (\nabla\times\mathbf{F})_z &= \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \\ &= \frac{k}{x^2+y^2} - \frac{2kx^2}{(x^2+y^2)^2} + \\ &\quad + \frac{k}{x^2+y^2} - \frac{2ky^2}{(x^2+y^2)^2} \\ &= \frac{2k}{x^2+y^2} - \frac{2k(x^2+y^2)}{(x^2+y^2)^2} \\ &= \frac{2k}{x^2+y^2} - \frac{2k}{(x^2+y^2)} \\ &= 0 \ \text{ provided } r \neq 0 \ . \end{aligned}$$

Notice that the final two terms can be considered to cancel only away from $r = 0$, where their denominators vanish and the terms themselves are infinite.

(**b**)

Consider the loop integral $\oint \mathbf{F} \cdot d\mathbf{l}$ counterclockwise around a circular path of fixed radius R. What is the value of this integral?

Solution:

Working in cylindrical coordinates,

$$\begin{aligned}\oint \mathbf{F} \cdot d\mathbf{r} &= 2\pi R F_\varphi(R) \\ &= 2\pi k > 0 \ .\end{aligned}$$

(**c**)

Can $\mathbf{F}$ be derived from a single-valued potential U, *e.g.* $\mathbf{F} = -\nabla U$ where $U = U(r, \varphi, z)$? Why or why not?

Solution:

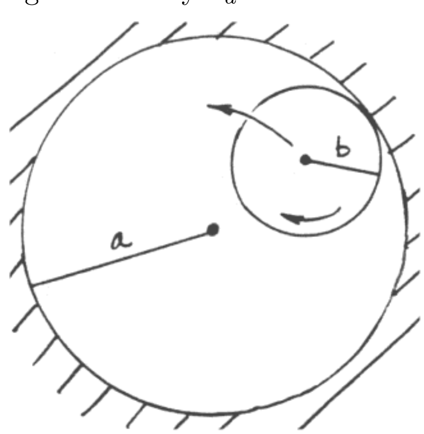
If it were true that $\mathbf{F} = -\nabla U$, then

$$\begin{aligned}\oint \mathbf{F} \cdot d\mathbf{r} &= \oint (-\nabla U) \cdot d\mathbf{r} \\ &= -U_0 + U_0 \\ &= 0 \ ,\end{aligned}$$

where U_0 is the single value of U at the starting (and ending) point of the loop integral. This condition is not satisfied from (**b**), so it is *not* true that $\mathbf{F} = -\nabla U$ even though, as shown in part (**a**), $\nabla \times \mathbf{F} = 0$ at finite r.

6.

A hoop of radius b and mass m rolls without slipping within a circular hole of radius $a > b$. About the center of the hole, the point of contact has a uniform angular velocity ω_a.



(**a**)

Find the angular velocity ω_b of the hoop about *its own* center (magnitude and direction).

Solution:

Imagine for a moment that the hoop is moving *while fully* slipping, *i.e.* imagine that the same point on the hoop always makes contact with the hole. In one counterclockwise (CCW) revolution of the point of contact, the hoop would also undergo one CCW revolution.

Now back to the real problem. Take the hoop to be rolling *without* slipping. In one revolution of the point of contact, the point of contact moves a distance $2\pi a$ along the circumference of the hole. Since the circumference of the hoop is smaller, $2\pi b$ instead of $2\pi a$, the fact that it is rolling causes the hoop to rotate *clockwise* by an *extra* number of revolutions equal to $2\pi a/2\pi b$.

Putting the above arguments into an equation, considering the CCW direction to be positive,

$$\omega_b = +\omega_a - \frac{a}{b}\omega_a \ .$$

Here the first term on the RHS is the angular velocity that would result from *full slipping*, while the second term results from the extra revolutions of the hoop due to rolling *without slipping*. Simplifying,

$$\omega_b = -(\tfrac{a}{b} - 1)\omega_a \ .$$

(**b**)

Calculate the kinetic energy T of the hoop.

Solution:

This is a straightforward application of a decomposition rule: the total kinetic energy T is the sum of the kinetic energy $T_{\text{of CM}}$ *of* the CM and the kinetic energy $T_{\text{wrt CM}}$ *with respect to* the CM. Applying it,

$$\begin{aligned}T &= T_{\text{of CM}} + T_{\text{wrt CM}} \\ &= \tfrac{1}{2}m(a-b)^2\omega_a^2 + \tfrac{1}{2}mb^2\omega_b^2 \\ &= \tfrac{1}{2}m\omega_a^2\big((a-b)^2 + b^2(\tfrac{a}{b} - 1)^2\big) \\ &= m\omega_a^2(a-b)^2 \ .\end{aligned}$$

In the second line above, we used the fact that the radius of the circle described by the motion of the CM of the hoop is $a - b$.

(**c**)
Obtain the angular momentum L (magnitude and direction) of the hoop relative to the center of the hole.

Solution:
Again this is a straightforward application of a decomposition rule, this time for angular momentum instead of kinetic energy. Applying it, taking the positive direction of $\mathbf{L}$ to be out of the paper,

$$\begin{aligned} L &= L_{\text{of CM}} + L_{\text{wrt CM}} \\ &= (a-b)^2\omega_a + mb^2\omega_b \\ &= m\omega_a\big((a-b)^2 - b^2(\tfrac{a}{b} - 1)\big) \\ &= m\omega_a(a^2 - 2ab + b^2 - ab + b^2) \\ &= m\omega_a(a^2 - 3ab + 2b^2) \\ &= m\omega_a(a-b)(a-2b) \ . \end{aligned}$$

(**d**)
Consider your answers to (**b**) and (**c**) in the limit $b \to a$. You should find that both T and L vanish. Is this reasonable? Why or why not?

Solution:
Both T and L contain the factor $(a - b)$, and so they vanish as b approaches a. (Amusingly, L also vanishes when $b = a/2$.) This is perfectly reasonable. Both T and L *must* vanish in the limit $a \to b$; in this limit, the point of contact rotates without the hoop moving at all!

7.
Consider a spherically symmetric distribution $\rho(r)$ of mass density. If the gravitational acceleration, or "gravitational field vector" $\mathbf{g}$, is known to be independent of the radial coordinate r within a spherical volume, find $\rho(r)$ to within a multiplicative constant.

Solution:
Remember: The gravitational acceleration due to the mass of a spherical shell vanishes if you are inside the shell. But if you are outside the shell, then the acceleration is the same as if the mass were all concentrated at the center. In our case, this means that you can find the acceleration g at any distance r from the origin by considering only the mass lying at distance less than r from the origin:

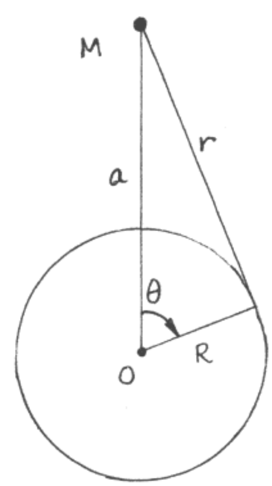
$$\begin{aligned} g(r) &= \frac{GM_{<r}}{r^2} \\ &= \frac{G}{r^2}\int_0^r \rho(r')\,d^3x' \\ &= \frac{4\pi G}{r^2}\int_0^r \rho(r')r'^2\,dr' \ . \end{aligned}$$

Now, if we want g to be a constant, then we need the integral in this equation to be proportional to r^2 (to cancel the $1/r^2$). If the integral is proportional to r^2, then the integrand is proportional to r. So

$$\begin{aligned} r^2\rho(r) &= kr \qquad \text{where k is a constant} \\ \rho &= k/r \ . \end{aligned}$$

Therefore ρ is proportional to $1/r$. (The constant k is $g/2\pi G$, in case you're interested.)

8.
Consider a point mass that lies outside a spherical surface. Let $\phi(\mathbf{r})$ be the gravitational potential due to the point mass.



Show that the average value of ϕ taken over the spherical surface is the same as the value of ϕ at the center of the sphere. [Since the potential due to an arbitrary mass distribution is the sum of potentials due to point masses, this statement is also true for the gravitational potential due to an arbitrary mass distribution lying outside a spherical surface.]

Solution:
Let a be the distance from the origin to the point mass, and R the radius of the sphere. The average of ϕ over the surface of the sphere is the integral over the whole surface divided by the surface area: $\langle\phi\rangle = \frac{1}{4\pi R^2}\int_{\text{sphere}}(-GM/r)\,dA$. By the law of cosines,

$$r = \sqrt{a^2 + R^2 - 2aR\cos\theta}\ ,$$

and of course the surface area element of the sphere is $dA = R^2\sin\theta\,d\theta\,d\varphi$, where φ is the "longitudinal" angle around the surface of the sphere.

$$\begin{aligned}
\langle\phi\rangle &= -\frac{GM}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} \frac{\sin\theta\,d\theta}{\sqrt{a^2+R^2-2aR\cos\theta}} \\
u &\equiv \cos\theta \\
\langle\phi\rangle &= -\tfrac{1}{2}GM\int_{-1}^{1}\frac{du}{\sqrt{a^2+R^2-2aRu}} \\
&= \frac{GM}{2aR}\left[\sqrt{a^2+R^2-2aRu}\right]_{-1}^{1} \\
&= -\frac{GM}{2aR}\left(\sqrt{(a+R)^2}-\sqrt{(a-R)^2}\right) \\
&= -\frac{GM}{2aR}\,2R \\
&= -\frac{GM}{a}\ .
\end{aligned}$$

$-GM/a$ is the value of the potential at the center of the sphere, so we've shown what we intended to.

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

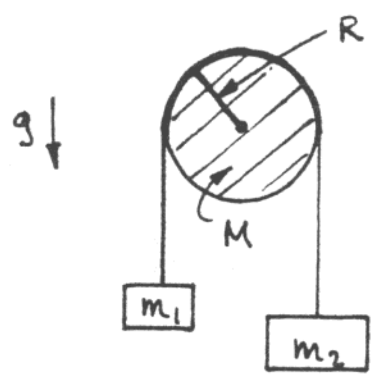
ASSIGNMENT 3

Reading:
105 Notes 4.1-4.6, 5.1-5.3.
Hand & Finch 1.4, 2.1-2.9, 1.10-1.11

1.
Use the calculus of variations to show that the shortest distance between two points in three-dimensional space is a straight line.

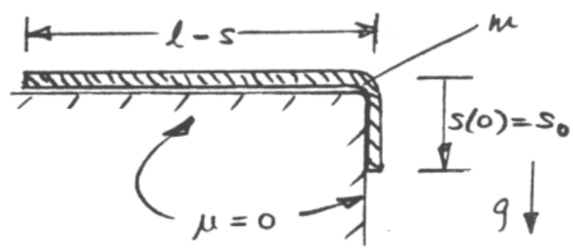
2.
Use the calculus of variations to obtain the function $\phi(\theta)$ describing the "great circle" path of minimum length on the surface of a sphere. This path connects spherical polar coordinates (θ_1, ϕ_1) with (θ_2, ϕ_2), in the general case where $\theta_1 \neq \theta_2$ and $\phi_1 \neq \phi_2$. Leave your answer in the form of an integral equation. [*Hint*: consider θ to be a "label" (like time t), and ϕ to be a coordinate (like $q(t)$).]

3.
Set up and solve the Euler-Lagrange equation for the Atwood machine, released from rest. (Two weights $m_1 < m_2$ are suspended via a massless string that is supported by a pulley in the form of a disk of radius R and mass M. The string moves without slipping on the pulley.)



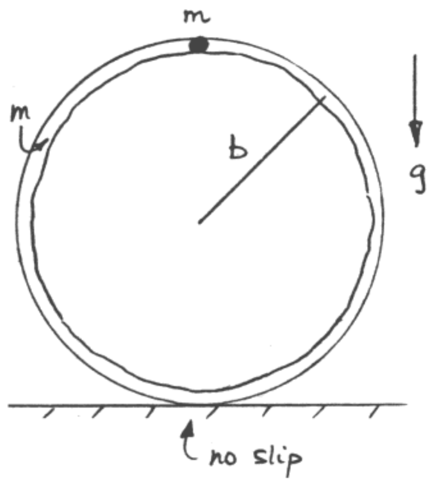
Use the height $y(t)$ of the smaller mass as the generalized coordinate.

4.
A chain of mass m and length l lies on a frictionless table. Initially the chain is at rest, with a length $s = s_0$ of the chain hanging off the table's end. This causes the chain to fall off the table. The part of the chain that remains on the table is straight, not coiled.



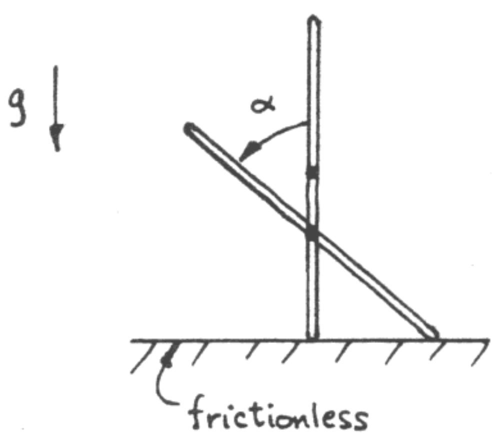
Using the Euler-Lagrange equation with s as the generalized coordinate, calculate the motion of the chain (before it falls off completely). Assume that the chain remains in contact with the corner and end of the table as shown (even though this in fact is true only for the early part of the motion).

5.
A bead of mass m moves inside a thin hoop-shaped pipe of average radius b, also of mass m. The pipe has a frictionless interior, so that the bead moves freely within the circumference of the hoop. But the coefficient of friction between the floor and the pipe's exterior is large, so that the hoop rolls on the floor without slipping.



The bead is released from rest at the top of the hoop. When the bead has fallen halfway to the floor, how far to the side will the hoop have moved?

6.
At $t = 0$, a thin uniform stick, resting on a frictionless floor, is erect and motionless. Let α represent the angle it makes with the vertical (initially $\alpha = 0$).



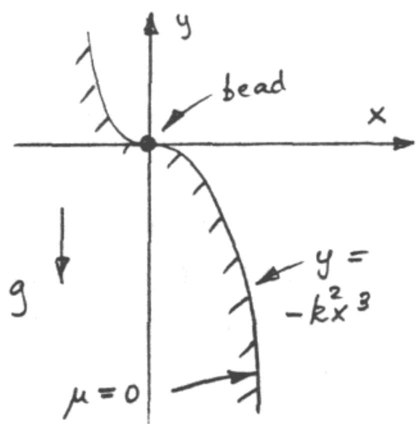
(**a**)
Use the Euler-Lagrange equation to obtain an equation relating $\ddot{\alpha}$ to α and $\dot{\alpha}$.
(**b**)
Because the floor is frictionless, total mechanical energy is conserved in this problem. Use this fact to relate $\dot{\alpha}$ to α.
(**c**)
Use the result of (**b**) to eliminate $\dot{\alpha}$ from your answer to (**a**), thereby obtaining an equation relating $\ddot{\alpha}$ to α alone. This equation should be valid for all values of α.
(**d**)
In the limit $\alpha \ll 1$, solve the result of (**c**) for the motion $\alpha(t)$.

7.
Continue to consider the stick in the previous problem. Use the method of Lagrange undetermined multipliers to find the force of constraint exerted by the floor on the stick, at the instant before the side of the stick impacts the floor.

8.
A bead moves under the influence of gravity on a frictionless surface described by

$$y = -k^2 x^3 ,$$

where k is a constant, and x and y are the horizontal and vertical coordinates.



The bead is released from rest at the origin. Use the method of Lagrange undetermined multipliers to solve for the coordinate $x = x_0$ at which it leaves the surface.

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO PROBLEM SET 3

Solutions by J. Barber and T. Bunn

Reading:
105 Notes 4.1-4.6, 5.1-5.3.
Hand & Finch 1.4, 2.1-2.9, 1.10-1.11

1.
Use the calculus of variations to show that the shortest distance between two points in three-dimensional space is a straight line.

Solution:
The distance from point 1 to point 2 along a curve $(x, y(x), z(x))$ is just

$$\begin{aligned}\ell &= \int_1^2 \sqrt{dx^2 + dy^2 + dz^2} \\ &= \int_1^2 \sqrt{1 + y'^2 + z'^2}\, dx\end{aligned}$$

where y' means dy/dx.

The variation of this integral is zero (and so the integral is an extremum) when the Euler-Lagrange equations

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y}$$

and

$$\frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) = \frac{\partial f}{\partial y}$$

are satisfied. In our case, these equations are

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0$$

and

$$\frac{d}{dx}\left(\frac{z'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0$$

So the two terms in the parentheses are constants. Call them p and q respectively. Then our two equations become

$$p^2 + (p^2 - 1)y'^2 + p^2 z'^2 = 0$$

and

$$q^2 + q^2 y'^2 + (q^2 - 1)z'^2 = 0$$

Solving for y' and z' gives

$$y' = \frac{p}{\sqrt{1 - p^2 - q^2}}\ , \quad z' = \frac{q}{\sqrt{1 - p^2 - q^2}}$$

So y' and z' are just constants. But that means that $y(x)$ and $z(x)$ are just ordinary linear equations. So our curve is a straight line.

2.
Use the calculus of variations to obtain the function $\phi(\theta)$ describing the "great circle" path of minimum length on the surface of a sphere. This path connects spherical polar coordinates (θ_1, ϕ_1) with (θ_2, ϕ_2), in the general case where $\theta_1 \neq \theta_2$ and $\phi_1 \neq \phi_2$. Leave your answer in the form of an integral equation. [*Hint*: consider θ to be a "label" (like time t), and ϕ to be a coordinate (like $q(t)$).]

Solution:
In spherical coordinates, an infinitesimal displacement $\vec{dl}$ can be written as:

$$\vec{dl} = dr\ \hat{r} + r\ d\theta\ \hat{\theta} + r\ \sin\theta\ d\phi\ \hat{\phi}$$

In the context of this problem, however, we will be considering a path on the surface of a sphere, so $dr = 0$, and $r = R =$ constant. Also,

$$d\phi = \frac{d\phi}{d\theta} d\theta = \phi' d\theta\ .$$

Hence, the total path length on the surface of the sphere between θ_1 and θ_2 is

$$\begin{aligned} l &= \int_{\theta_1}^{\theta_2} \left|\vec{dl}\right| \\ &= R \int_{\theta_1}^{\theta_2} d\theta \sqrt{1 + \sin^2\theta\ \phi'^2}\end{aligned}$$

Calling the integrand $\mathcal{L}(\phi,\phi',\theta)$, we apply the Euler-Lagrange equation:

$$\frac{d}{d\theta}\frac{\partial \mathcal{L}}{\partial \phi'} = \frac{\partial \mathcal{L}}{\partial \phi}$$
$$\frac{d}{d\theta}\left(\frac{\sin^2\theta\,\phi'}{\sqrt{1+\sin^2\theta\,\phi'^2}}\right) = 0$$
$$\frac{\sin^2\theta\,\phi'}{\sqrt{1+\sin^2\theta\,\phi'^2}} = c$$

where c is a constant of integration. Solving for ϕ' yields

$$\phi' = \frac{c}{\sin\theta\sqrt{\sin^2\theta - c^2}}\,.$$

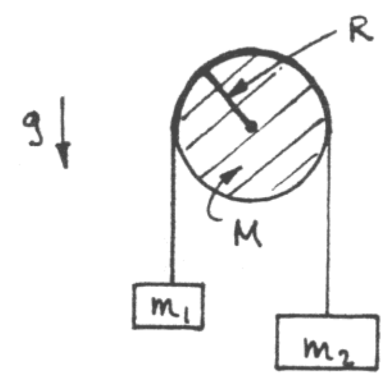
Both sides of this expression can be integrated from θ_1 to θ to obtain

$$\phi(\theta) = \int_{\theta_1}^{\theta} \frac{c}{\sin\theta'\sqrt{\sin^2\theta' - c^2}}\,d\theta' + \phi_1$$

where c is chosen so that $\phi(\theta_2) = \phi_2$.

3.

Set up and solve the Euler-Lagrange equation for the Atwood machine, released from rest. (Two weights $m_1 < m_2$ are suspended via a massless string that is supported by a pulley in the form of a disk of radius R and mass M. The string moves without slipping on the pulley.)



Use the height $y(t)$ of the smaller mass as the generalized coordinate.

Solution:

Let's choose as our generalized coordinate y, the distance of mass m_1 above its starting point. Then $-y$ is the distance of m_2 above its starting point, and $\theta = y/R$ is the angle through which the wheel has rotated. The Lagrangian is

$$\begin{aligned}\mathcal{L} &= T - V\\ &= \tfrac{1}{2}(m_1+m_2)\dot{y}^2 + \tfrac{1}{2}I\dot{\theta}^2 - (m_1-m_2)gy\end{aligned}$$

The moment of inertia of a disc is $I = \frac{1}{2}MR^2$, so

$$\mathcal{L} = \tfrac{1}{2}(m_1+m_2)\dot{y}^2 + \tfrac{1}{4}M\dot{y}^2 - (m_1-m_2)gy$$

Let's define $\mu = m_1 + m_2 + \frac{1}{2}M$ and $\Delta m = m_1 - m_2$. Then $\mathcal{L} = \frac{1}{2}\mu\dot{y}^2 - \Delta m g y$, and the Euler-Lagrange equation $\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right)$ gives

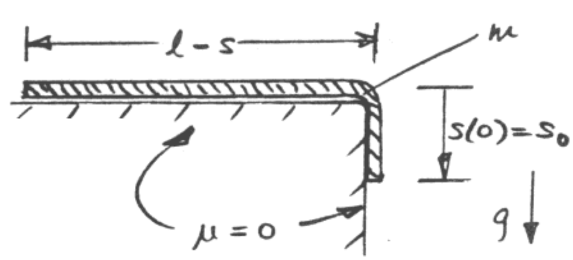
$$\mu\ddot{y} = -\Delta m g$$

The acceleration is constant: $a = -\frac{\Delta m}{\mu}g$. So

$$y = -\frac{1}{2}\frac{\Delta m}{\mu}gt^2 = \left(\frac{m_2 - m_1}{2m_1 + 2m_2 + M}\right)gt^2$$

4.

A chain of mass m and length l lies on a frictionless table. Initially the chain is at rest, with a length $s = s_0$ of the chain hanging off the table's end. This causes the chain to fall off the table. The part of the chain that remains on the table is straight, not coiled.



Using the Euler-Lagrange equation with s as the generalized coordinate, calculate the motion of the chain (before it falls off completely). Assume that the chain remains in contact with the corner and end of the table as shown (even though this in fact is true only for the early part of the motion).

Solution:

Let's take the zero point of the potential energy to be the surface of the table. That way, only that portion of the chain which is hanging contributes to the potential energy. The Lagrangian is

$$\begin{aligned}\mathcal{L} &= T - U \\ &= \frac{1}{2}m\dot{s}^2 + \frac{mg}{2l}s^2\end{aligned}$$

Applying the Euler-Lagrange equation:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{s}}\right) &= \frac{\partial \mathcal{L}}{\partial s} \\ \ddot{s} &= \frac{g}{l}s\end{aligned}$$

The general solution of which is

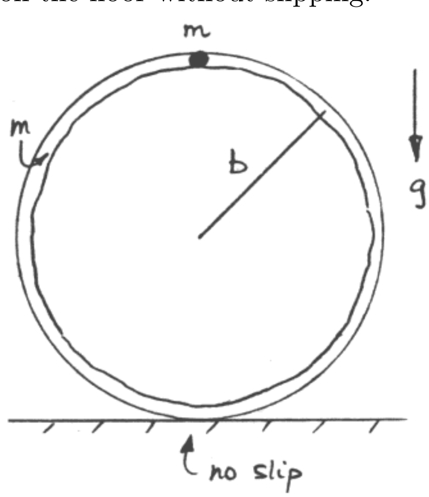
$$s(t) = A\sinh\sqrt{\frac{g}{l}}t + B\cosh\sqrt{\frac{g}{l}}t$$

where A and B are constants. We determine A and B using our boundary conditions, s(0) $= s_o$ and $\dot{s}(0) = 0$, to get

$$s(t) = s_o\cosh\sqrt{\frac{g}{l}}t$$

5.

A bead of mass m moves inside a thin hoop-shaped pipe of average radius b, also of mass m. The pipe has a frictionless interior, so that the bead moves freely within the circumference of the hoop. But the coefficient of friction between the floor and the pipe's exterior is large, so that the hoop rolls on the floor without slipping.



The bead is released from rest at the top of the hoop. When the bead has fallen halfway to the floor, how far to the side will the hoop have moved?

Solution:

Let's take as our generalized coordinates θ, the angle of the bead from the top of the hoop, and x, the sideways distance that the center of the hoop has moved from its starting point. We'll assume that the bead falls clockwise, for which we'll define θ to be positive, and we'll take x to be positive in the *left*-hand direction. The kinetic energy is

$$\begin{aligned}T &= T_{\text{hoop}} + T_{\text{bead}} \\ T_{\text{hoop}} &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mb^2\left(\frac{\dot{x}}{b}\right)^2 \\ &= m\dot{x}^2 \\ T_{\text{bead}} &= \frac{1}{2}m\left(\left(-\dot{x} + b\dot{\theta}\cos\theta\right)^2 + \left(-b\dot{\theta}\sin\theta\right)^2\right) \\ &= \frac{m}{2}\left(\dot{x}^2 + b^2\dot{\theta}^2 - 2b\dot{x}\dot{\theta}\cos\theta\right) \\ T &= m\left(\frac{3}{2}\dot{x}^2 + \frac{1}{2}b^2\dot{\theta}^2 - b\dot{x}\dot{\theta}\cos\theta\right)\end{aligned}$$

If we take the zero of the potential energy to be at the center of the hoop, then only the bead has potential energy.

$$U = mgb\cos\theta$$

Thus the Lagrangian is

$$\begin{aligned}\mathcal{L} &= T - U \\ &= m\left(\frac{3}{2}\dot{x}^2 + \frac{1}{2}b^2\dot{\theta}^2 - b\cos\theta(mg + \dot{x}\dot{\theta})\right)\end{aligned}$$

Applying the Euler-Lagrange equation to the coordinate x:

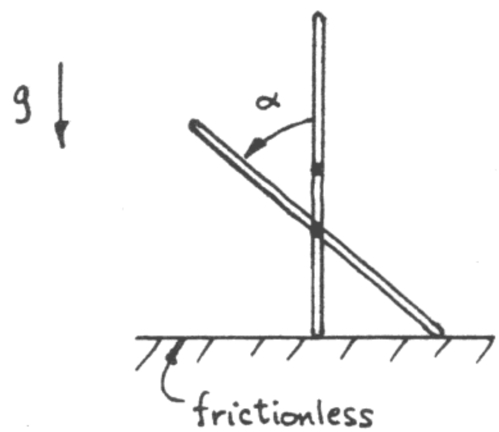
$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) &= \frac{\partial \mathcal{L}}{\partial x} \\ \frac{d}{dt}\left(3\dot{x} - b\dot{\theta}\cos\theta\right) &= 0 \\ 3\dot{x} - b\dot{\theta}\cos\theta &= c_1 \quad \text{(a constant)}\end{aligned}$$

By the initial conditions, $\dot{x}(0) = \dot{\theta}(0) = 0$, $c_1 = 0$.

$$3\dot{x} - b\dot{\theta}\cos\theta = 0$$
$$\frac{d}{dt}(3x - b\sin\theta) = 0$$
$$3x - b\sin\theta = c_2 \quad (c_2 = 0 \text{ also})$$
$$x = \frac{1}{3}b\sin\theta$$

At $\theta = \frac{\pi}{2}$, $x = \frac{b}{3}$. The hoop is displaced in the opposite direction from that of the bead.

6.
At $t = 0$, a thin uniform stick, resting on a frictionless floor, is erect and motionless. Let α represent the angle it makes with the vertical (initially $\alpha = 0$).



(**a**)
Use the Euler-Lagrange equation to obtain an equation relating $\ddot{\alpha}$ to α and $\dot{\alpha}$.
Solution:
The height of the center of mass is $y = \frac{1}{2}l\cos\alpha$, so the kinetic energy (including both translational and rotational terms) is $T = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\dot{\alpha}^2$. The moment of inertia of a thin stick is $I = \frac{1}{12}ml^2$, so $T = \frac{1}{24}ml^2\dot{\alpha}^2(1 + 3\sin^2\alpha)$. The potential energy is $V = mgy = \frac{1}{2}mgl\cos\alpha$. So $\mathcal{L}$ is

$$\mathcal{L} = \tfrac{1}{24}ml^2\dot{\alpha}^2(1 + 3\sin^2\alpha) - \tfrac{1}{2}mgl\cos\alpha$$

The Euler-Lagrange equation for α is

$$\frac{1}{4}ml^2\dot{\alpha}^2\sin\alpha\cos\alpha + \frac{1}{2}mgl\sin\alpha =$$
$$\frac{1}{12}ml^2\left(\ddot{\alpha}(1 + 3\sin^2\alpha) + 6\dot{\alpha}^2\sin\alpha\cos\alpha\right)$$

Solving this for $\ddot{\alpha}$ yields

$$\ddot{\alpha} = \frac{6\frac{g}{l}\sin\alpha - 3\dot{\alpha}^2\sin\alpha\cos\alpha}{1 + 3\sin^2\alpha}$$

(**b**)
Because the floor is frictionless, total mechanical energy is conserved in this problem. Use this fact to relate $\dot{\alpha}$ to α.
Solution:

$$E = \frac{1}{24}ml^2\dot{\alpha}^2(1 + 3\sin^2\alpha) + \frac{1}{2}mgl\cos\alpha$$
$$= \frac{1}{2}mgl = E_{\text{initial}}$$
$$\dot{\alpha}^2 = \frac{12g}{l}\frac{1 - \cos\alpha}{1 + 3\sin^2\alpha}$$

(**c**)
Use the result of (**b**) to eliminate $\dot{\alpha}$ from your answer to (**a**), thereby obtaining an equation relating $\ddot{\alpha}$ to α alone. This equation should be valid for all values of α.
Solution:
Inserting the answer from (**b**) into the result of (**a**), and simplifying, yields

$$\ddot{\alpha} = \frac{6g}{l}\sin\alpha\left(\frac{1 + 3(1 - \cos\alpha)^2}{(1 + 3\sin^2\alpha)^2}\right)$$

(**d**)
In the limit $\alpha \ll 1$, solve the result of (**c**) for the motion $\alpha(t)$.
Solution:
When $\alpha \ll 1$, we keep only terms to first order in α. In that limit, the expression from (**c**) becomes:

$$\ddot{\alpha} \approx \frac{6g}{l}\alpha$$

If we take our initial conditions to be $\alpha(0) = \alpha_o \ll 1$ and $\dot{\alpha}(0) = 0$, then the solution to this is

$$\alpha(t) = \alpha_o\cosh\sqrt{\frac{6g}{l}}t$$

7.
Continue to consider the stick in the previous problem. Use the method of Lagrange undetermined multipliers to find the force of constraint exerted by the floor on the stick, at the instant before the side of the stick impacts the floor.
Solution:
Let v be the height of the bottom point of the stick from the floor. Because the stick's lower end is always in contact with the ground, we have the constraint that $v = 0$. The height of the CM is now $y = v + \frac{l}{2}\cos\alpha$.

$$\begin{aligned} T &= \frac{m}{2}\dot{y}^2 + \frac{1}{2}\left(\frac{1}{12}ml^2\right)\dot{\alpha}^2 \\ &= \frac{m}{8}l^2\dot{\alpha}^2\left(\frac{1}{3}+\sin^2\alpha\right) + \frac{m}{2}\dot{v}^2 - \frac{ml}{2}\dot{v}\dot{\alpha}\sin\alpha \\ V &= mgy \\ &= mg\left(v + \frac{1}{2}l\cos\alpha\right) \end{aligned}$$

From the equation of constraint, we know that $g_v = 1$. Applying the Euler-Lagrange equation to the v coordinate:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{v}}\right) &= \frac{\partial\mathcal{L}}{\partial v} + g_v\lambda \\ \frac{d}{dt}\left(m\dot{v} - \frac{ml}{2}\dot{\alpha}\sin\alpha\right) &= -mg + F_c \\ m\ddot{v} - \frac{ml}{2}\left(\ddot{\alpha}\sin\alpha + \dot{\alpha}^2\cos\alpha\right) &= -mg + F_c\ , \end{aligned}$$

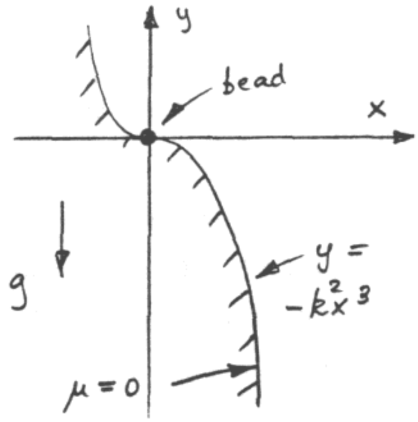
where $F_c = g_v\lambda$ is the generalized force of constraint. Using the fact that $\ddot{v} = \dot{v} = v = 0$, and the expression for $\ddot{\alpha}$ from **6(c)**, yields

$$\begin{aligned} F_c &= mg - \frac{ml}{2}\ddot{\alpha}|_{\alpha=\frac{\pi}{2}} \\ &= mg - \frac{ml}{2}\left(\frac{3}{2}\frac{g}{l}\right) \\ &= \frac{1}{4}mg \quad \text{(upward)} \end{aligned}$$

8.
A bead moves under the influence of gravity on a frictionless surface described by

$$y = -k^2x^3\ ,$$

where k is a constant, and x and y are the horizontal and vertical coordinates.



The bead is released from rest at the origin. Use the method of Lagrange undetermined multipliers to solve for the coordinate $x = x_0$ at which it leaves the surface.
Solution:

$$\begin{aligned} \mathcal{L} &= T - U \\ &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy \end{aligned}$$

Constraint:

$$\begin{aligned} y + k^2x^3 &= 0 \ \ \text{(Eq. 3)} \\ dy + 3k^2x^2dx &= 0 \\ \Rightarrow\ g_y\ =\ 1,\ g_x\ &=\ 3k^2x^2 \end{aligned}$$

Applying the Euler-Lagrange equations:

$$\begin{aligned} m\ddot{y} &= -mg + \lambda \quad \text{(Eq. 1)} \\ m\ddot{x} &= 3k^2x^2\lambda \quad\ \ \text{(Eq. 2)} \end{aligned}$$

Use (Eq. 3) to eliminate $\ddot{y}$ from (Eq. 1):

$$\begin{aligned} \dot{y} &= -3k^2x^2\dot{x} \\ \ddot{y} &= -6k^2x\dot{x}^2 - 3k^2x^2\ddot{x} \\ 3k^2x(x\ddot{x} + 2\dot{x}^2) &= g - \frac{\lambda}{m} \quad \text{(Eq. 1}'\text{)} \end{aligned}$$

Use (Eq. 2) to eliminate $\ddot{x}$ from (Eq. 1′):

$$3k^2x\left(3k^2x^3\frac{\lambda}{m} + 2\dot{x}^2\right) = g - \frac{\lambda}{m} \quad \text{(Eq. 1}''\text{)}$$

The bead loses contact when $\lambda = 0$. Set $\lambda = 0$ in (Eq. $1''$), to get:

$$6k^2 x \dot{x}^2|_{x_1=\text{breakaway}} = g \quad \text{(Eq. 4)}$$

Use energy conservation, $\dot{x}^2 + \dot{y}^2 = -2gy$, and (Eq. 3) to solve for x in terms of $\dot{x}$:

$$\dot{x}^2 + \left(-3k^2x^2\dot{x}\right)^2 = -2g(-k^2x^3)$$
$$\dot{x}^2 = \frac{2gk^2x^3}{1+9k^4x^4} \quad \text{(Eq. 5)}$$

Substitute (Eq. 5) in (Eq. 4):

$$6k^2x\frac{2gk^2x^3}{1+9k^4x^4}|_{x_1} = g$$
$$x_1 = \frac{1}{k3^{\frac{1}{4}}}$$

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

ASSIGNMENT 4

Reading:
105 Notes 6.1-6.2, 3.1-3.3
Hand & Finch 3.1-3.3

1.
Generalize the Euler equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\partial \mathcal{L}}{\partial y}$$

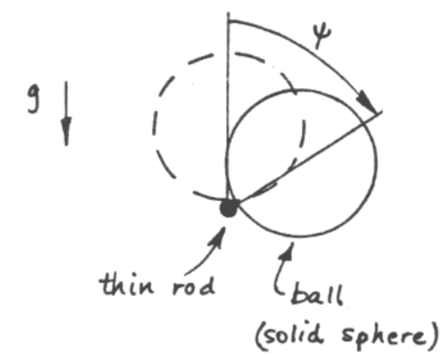
to the case in which $\mathcal{L}$ is a function of t, y, $\dot{y}$, *and* $\ddot{y}$. Derive the new Euler equation for this case. Assume that $y(t_1)$, $y(t_2)$, *and* $\dot{y}(t_1)$, $\dot{y}(t_2)$ are not varied, *i.e.* both the value and the slope of y are fixed at each endpoint.
[*Hint*: Compared to the derivation of the usual Euler equation, when you calculate the variation of the action J with the parameter α, you will have an extra term in the integrand. Integrate that term by parts *twice*.]

2.
A bead moves in a constant gravitational field $\mathbf{a} = \hat{\mathbf{x}}g$ with an initial velocity $|\mathbf{v}| = v_0$, where g and v_0 are positive constants. It is constrained to slide along a frictionless wire which has an unknown shape $y(x)$. (Notice that $\hat{\mathbf{x}}$ points down and $\hat{\mathbf{y}}$ points to the right in this problem.)
(**a**)
Show that the shape $y(x)$ which minimizes the bead's transit time between two fixed points $(0,0)$ and (X,Y) is given by a set of parametric equations

$$x = x_0 + a(1 - \cos\phi)$$
$$y = y_0 + b(\phi - \sin\phi)\ ,$$

where ϕ is the parameter. This is the famous *brachistochrone problem*. The solution is a *cycloid* – the path of a dot painted on a rolling wheel.
(**b**)
In terms of v_0, g, X, and Y, what are the values of the constants x_0, y_0, a, and b which yield the optimal trajectory? Give definite answers where you can; otherwise provide equations which, if solved, would yield those values. [*Hint*: see Hand & Finch, problems 2.9 and 2.10.]

3.
Starting from a vertical position at rest, a solid ball resting on top of a thin rod falls off.



While in contact with the rod, the ball rolls without slipping. Using the method of Lagrange undetermined multipliers, find the angle ψ at which the ball leaves the rod ($\psi \equiv 0$ initially).

4.
Consider a simple, plane pendulum consisting of a mass m attached to a string of length l. Only small oscillations need be considered. After the pendulum is set into motion, the length of the string is shortened at a constant rate $dl/dt = -\alpha$, where $\alpha > 0$. (The string is pulled through a small hole located at a constant position, so the pendulum's suspension point remains fixed.) Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy of the pendulum, and discuss the conservation of energy for the system.

5.
A particle of mass m and velocity $\mathbf{v}_1$ leaves a semi-infinite space $z < 0$, where the potential energy is a constant U_1, and enters the remaining space $z > 0$, where the potential is a constant U_2.
(**a**)
Use symmetry arguments to find two constants of the motion.
(**b**)
Use these two constants to obtain the new velocity $\mathbf{v}_2$.

6.
The Lagrangian for a (physically interesting) system is

$$\mathcal{L}(\varphi, \dot{\varphi}, \theta, \dot{\theta}, \psi, \dot{\psi}, t) = \tfrac{1}{2} I (\dot{\varphi}^2 \sin^2 \theta + \dot{\theta}^2) + \\ + \tfrac{1}{2} I_3 (\dot{\varphi} \cos \theta + \dot{\psi})^2 - mgh \cos \theta \ ,$$

where (φ, θ, ψ) are Euler angles and (I, I_3, mgh) are constants.
(**a**)
Find two cyclic coordinates and obtain the two corresponding conserved canonically conjugate momenta.
(**b**)
Find a third constant of the motion.
(**c**)
Using the results of (**a**) and (**b**), express $\dot{\theta}^2$ as a function only of θ and constants.

7.
The interaction Lagrangian for a system consisting of a relativistic test particle of mass m and charge e moving in a static electromagnetic field is

$$\mathcal{L}(\mathbf{x}, \mathbf{v}, t) = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} + e\mathbf{v} \cdot \mathbf{A} - e\phi \ ,$$

where $\mathbf{x}$ is the particle's position, $\mathbf{v}$ is its velocity, $\phi(\mathbf{x})$ is the electrostatic potential ($\mathbf{E} = -\nabla\phi$), $\mathbf{A}$ is the (static) magnetic vector potential ($\mathbf{B} = \nabla \times \mathbf{A}$), and c is the speed of light.

(**a**)
Write down the canonical momenta (p_1, p_2, p_3) which are conjugate to the Cartesian coordinates (x_1, x_2, x_3).
(**b**)
Compute the Hamiltonian $\mathcal{H}(\mathbf{x}, \mathbf{v}, t)$.
(**c**)
Re-express $\mathcal{H}(\mathbf{x}, \mathbf{v}, t)$ as the function $\mathcal{H}(\mathbf{x}, \mathbf{p}, t)$.
(**d**)
Show that $\mathcal{H}$ is conserved. Is it equal to

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \ ,$$

the total (relativistic) energy of the test particle? Explain.

8.
Consider f and g to be any two continuous functions of the generalized coordinates q_i and canonically conjugate momenta p_i, as well as time:

$$f = f(q_i, p_i, t)$$
$$g = g(q_i, p_i, t) \ .$$

The *Poisson bracket* of f and g is defined by

$$[f, g] \equiv \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \ ,$$

where summation over i is implied. Prove the following properties of the Poisson bracket:
(**a**)

$$\frac{df}{dt} = [f, \mathcal{H}] + \frac{\partial f}{\partial t}$$

(**b**)

$$\dot{q}_i = [q_i, \mathcal{H}]$$

(**c**)

$$\dot{p}_i = [p_i, \mathcal{H}]$$

(**d**)

$$[p_i, p_j] = 0$$

(**e**)

$$[q_i, q_j] = 0$$

(**f**)

$$[q_i, p_j] = \delta_{ij} \ ,$$

where $\mathcal{H}$ is the Hamiltonian. If the Poisson bracket of two quantities is equal to unity, the quantities are said to be *canonically conjugate*. On the other hand, if the Poisson bracket vanishes, the quantities are said to *commute*.
(**g**)
Show that any quantity that does not depend explicitly on the time and that commutes with the Hamiltonian is a constant of the motion.

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SOLUTION TO PROBLEM SET 4
Solutions by J. Barber and T. Bunn

Reading:
105 Notes 6.1-6.2, 3.1-3.3
Hand & Finch 3.1-3.3

1.
Generalize the Euler equation

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{y}} = \frac{\partial\mathcal{L}}{\partial y}$$

to the case in which $\mathcal{L}$ is a function of t, y, $\dot{y}$, *and* $\ddot{y}$. Derive the new Euler equation for this case. Assume that $y(t_1)$, $y(t_2)$, *and* $\dot{y}(t_1)$, $\dot{y}(t_2)$ are not varied, *i.e.* both the value and the slope of y are fixed at each endpoint.
[*Hint*: Compared to the derivation of the usual Euler equation, when you calculate the variation of the action J with the parameter α, you will have an extra term in the integrand. Integrate that term by parts *twice*.]
Solution:

$$J = \int_{t_1}^{t_2} \mathcal{L}(y, \dot{y}, \ddot{y}, t)dt$$

As usual, suppose $y = y(t, \alpha)$, where α is a parameter, and suppose $y(t, \alpha = 0)$ minimizes J. Thus:

$$\frac{dJ}{d\alpha} = \int_{t_1}^{t_2}\left(\frac{\partial\mathcal{L}}{\partial y}\frac{dy}{d\alpha} + \frac{\partial\mathcal{L}}{\partial\dot{y}}\frac{d\dot{y}}{d\alpha} + \frac{\partial\mathcal{L}}{\partial\ddot{y}}\frac{d\ddot{y}}{d\alpha}\right)dt = 0$$

Integrate the second term once by parts:

$$\begin{aligned}&\int_{t_1}^{t_2}\frac{\partial\mathcal{L}}{\partial\dot{y}}\frac{d}{dt}\left(\frac{dy}{d\alpha}\right)dt \\ &= \frac{\partial\mathcal{L}}{\partial\dot{y}}\frac{dy}{d\alpha}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{dy}{d\alpha}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right)dt \\ &= 0 - \int_{t_1}^{t_2}\frac{dy}{d\alpha}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right)dt\ ,\end{aligned}$$

where the '0' follows from the endpoints being fixed. Now integrate the second term by parts twice:

$$\begin{aligned}&\int_{t_1}^{t_2}\frac{\partial\mathcal{L}}{\partial\ddot{y}}\frac{d\ddot{y}}{d\alpha}dt \\ &= \frac{\partial\mathcal{L}}{\partial\ddot{y}}\frac{d\dot{y}}{d\alpha}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\ddot{y}}\right)\frac{d}{dt}\left(\frac{dy}{d\alpha}\right)dt \\ &= \frac{\partial\mathcal{L}}{\partial\ddot{y}}\frac{d\dot{y}}{d\alpha}\Big|_{t_1}^{t_2} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\ddot{y}}\right)\frac{dy}{d\alpha}\Big|_{t_1}^{t_2} \\ &\quad + \int_{t_1}^{t_2}\frac{dy}{d\alpha}\frac{d^2}{dt^2}\left(\frac{\partial\mathcal{L}}{\partial\ddot{y}}\right)dt \\ &= 0 - 0 + \int_{t_1}^{t_2}\frac{dy}{d\alpha}\frac{d^2}{dt^2}\left(\frac{\partial\mathcal{L}}{\partial\ddot{y}}\right)dt\end{aligned}$$

What remains is:

$$\begin{aligned}\frac{dJ}{d\alpha} &= \int_{t_1}^{t_2}\left(\frac{\partial\mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{y}} + \frac{d^2}{dt^2}\frac{\partial\mathcal{L}}{\partial\ddot{y}}\right)\frac{dy}{d\alpha}dt \\ &= 0\end{aligned}$$

The only way the above can be true for arbitrary variation with α is for the quantity in parenthesis to be zero. In other words:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right) = \frac{\partial\mathcal{L}}{\partial y} + \frac{d^2}{dt^2}\left(\frac{\partial\mathcal{L}}{\partial\ddot{y}}\right)$$

It is easy to see what the pattern would be for a lagrangian depending on yet higher derivatives of y.

2.
A bead moves in a constant gravitational field $\mathbf{a} = \hat{\mathbf{x}}g$ with an initial velocity $|\mathbf{v}| = v_0$, where g and v_0 are positive constants. It is constrained to slide along a frictionless wire which has an unknown shape $y(x)$. (Notice that $\hat{\mathbf{x}}$ points down and $\hat{\mathbf{y}}$ points to the right in this problem.)

(**a**)
Show that the shape $y(x)$ which minimizes the bead's transit time between two fixed points $(0, 0)$ and (X, Y) is given by a set of parametric equations

$$x = x_0 + a(1 - \cos\phi)$$
$$y = y_0 + b(\phi - \sin\phi)\ ,$$

where ϕ is the parameter. This is the famous *brachistochrone problem*. The solution is a *cycloid* – the path of a dot painted on a rolling wheel.
Solution:
The transit time, starting from (X,Y) and ending up at (0,0), is given by:

$$T = \int_X^0 dt = \int_X^0 \frac{dl}{v} = \int_X^0 \frac{\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}}{\sqrt{2g}\sqrt{\frac{E}{mg} + x}}\, dx$$

where the denominator in the last step follows from conservation of energy, with $E = \frac{1}{2}m{v_o}^2 - mgX$. Calling the integrand $\mathcal{L}$, we apply the Euler equation:

$$\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) = \frac{\partial \mathcal{L}}{\partial y} = 0$$
$$\frac{d}{dx}\left(\frac{y'}{\sqrt{y'^2+1}\sqrt{\frac{E}{mg}+x}}\right) = 0$$
$$\frac{y'}{\sqrt{y'^2+1}\sqrt{\frac{E}{mg}+x}} = C \ \ \text{(a constant)}$$

To show that the above parametric equations for $x(\phi)$ and $y(\phi)$ are the solution to this D.E., we'll need the following:

$$y' = \frac{dy}{dx} = \frac{dy}{d\phi}\frac{d\phi}{dx} = \frac{b}{a}\frac{(1-\cos\phi)}{\sin\phi}$$

Substituting this into the D.E. yields:

$$\frac{b}{a}\frac{(1-\cos\phi)}{\sin\phi\sqrt{\frac{b^2}{a^2}\frac{(1-\cos\phi)^2}{\sin^2\phi}+1}\sqrt{\frac{E}{mg}+x_o+a(1-\cos\phi)}} = C$$
$$\frac{(1-\cos\phi)}{\sqrt{1-2\cos\phi+\cos^2\phi+\frac{a^2}{b^2}\sin^2\phi}\sqrt{\frac{E}{mg}+x_o+a(1-\cos\phi)}} = C$$

If $a = b$ and $x_o = -\frac{E}{mg}$, then the above is true for all ϕ, as it reduces to:

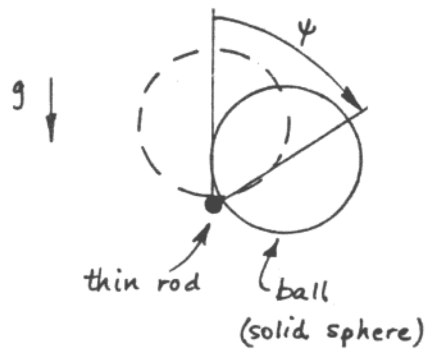
$$\frac{(1-\cos\phi)}{\sqrt{2(1-\cos\phi)}\sqrt{1-\cos\phi}\sqrt{a}} = \frac{1}{\sqrt{2a}} = C$$

which is a true statement.
(**b**)
In terms of v_0, g, X, and Y, what are the values of the constants x_0, y_0, a, and b which yield the optimal trajectory? Give definite answers where you can; otherwise provide equations which, if solved, would yield those values. [*Hint*: see Hand & Finch, problems 2.9 and 2.10.]
Solution:
Let's say that, at ϕ_i, (x,y)=(X,Y), and at ϕ_f (x,y)=(0,0). This, along with the conditions that a=b and $x_o = \frac{E}{mg}$ from part (a), yields the following equations:

$$X = -\frac{{v_o}^2}{2g} + X + a(1-\cos\phi_i)$$
$$Y = y_o + a(\phi_i - \sin\phi_i)$$
$$0 = -\frac{{v_o}^2}{2g} + X + a(1-\cos\phi_f)$$
$$0 = y_o + a(\phi_f - \sin\phi_f)$$

This is a set of four transcendental equations for four unknowns: a, y_o, ϕ_i, and ϕ_f.

3.
Starting from a vertical position at rest, a solid ball resting on top of a thin rod falls off.



While in contact with the rod, the ball rolls without slipping. Using the method of Lagrange undetermined multipliers, find the angle ψ at which the ball leaves the rod ($\psi \equiv 0$ initially).

Solution:
Let's pick as our coordinates ψ, the angle down from the y-axis that the ball has rolled (y points up), and r, the distance of the center of the sphere from the thin rod. (So we're working with the ordinary polar coordinates of the center of the sphere.) The equation of constraint is $r = R$, where R is the radius of the sphere. Then the kinetic energy is $T = \frac{1}{5}mR^2\dot{\psi}^2 + \frac{1}{2}mr^2\dot{\psi}^2 + \frac{1}{2}m\dot{r}^2$. (The first term represents the rotation of the sphere about its center, and the other two represent translations of the center of the sphere in the tangential and radial directions.) The potential energy is $V = mgr\cos\psi$. So

$$\mathcal{L} = \tfrac{1}{2}m\left(\tfrac{2}{5}R^2 + r^2\right)\dot{\psi}^2 + \tfrac{1}{2}m\dot{r}^2 - mgr\cos\psi$$

The Euler-Lagrange equation for r is

$$\ddot{r} - r\dot{\psi}^2 + g\cos\psi = \lambda$$

Use the equation of constraint to remove the $\ddot{r}$ from the above and replace r by the constant R: $\lambda = -R\dot{\psi}^2 + g\cos\psi$. We want to find the angle ψ_ℓ at which the force of constraint λ equals zero: $\dot{\psi}_\ell^2 = (g/R)\cos\psi_\ell$. Let's get rid of the $\dot{\psi}_\ell$. Energy conservation says

$$\tfrac{7}{10}mR^2\dot{\psi}^2 + mgR\cos\psi = mgR$$

Solve for $\dot{\psi}$: $\dot{\psi}^2 = \frac{10g}{7R}(1-\cos\psi)$. Substitute into the equation for ψ_ℓ, and you get

$$\cos\psi_\ell = \tfrac{10}{17}, \qquad \text{so} \qquad \psi_\ell \approx 54^\circ.$$

4.
Consider a simple, plane pendulum consisting of a mass m attached to a string of length l. Only small oscillations need be considered. After the pendulum is set into motion, the length of the string is shortened at a constant rate $dl/dt = -\alpha$, where $\alpha > 0$. (The string is pulled through a small hole located at a constant position, so the pendulum's suspension point remains fixed.) Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy of the pendulum, and discuss the conservation of energy for the system.

Solution:
The lagrangian is $\mathcal{L} = \frac{m}{2}\dot{\ell}^2 + \frac{m}{2}\ell^2\dot{\theta}^2 + mg\ell\cos\theta$. Use $\ell = \ell_0 - \alpha t$ to write this in terms of the variables $(\theta, \dot{\theta}, t)$:

$$\mathcal{L}(\theta,\dot{\theta},t) = \tfrac{1}{2}m\alpha^2 + \tfrac{1}{2}m(\ell_0-\alpha t)^2\dot{\theta}^2 + mg(\ell_0-\alpha t)\cos\theta$$

The momentum canonically conjugate to the coordinate θ is $p = \frac{d\mathcal{L}}{d\dot{\theta}} = m(\ell_0-\alpha t)^2\dot{\theta}$, so we can write

$$\mathcal{H}(\theta,p,t) = p\dot{\theta} - \mathcal{L}$$

$$= \tfrac{1}{2}m(\ell_0-\alpha t)^2\dot{\theta}^2 - \tfrac{1}{2}m\alpha^2 - mg(\ell_0-\alpha t)\cos\theta$$
$$= \frac{p^2}{2m(\ell_0-\alpha t)^2} - \tfrac{1}{2}m\alpha^2 - mg(\ell_0-\alpha t)\cos\theta$$

This is not the same as the total energy. (The energy would have a + sign in the second term.) The rules about hamiltonians say that $\mathcal{H} = E$ if the generalized coordinates are related to ordinary cartesian coordinates in a way that doesn't depend explicitly on t, and if there are no velocity-dependent potentials. That first condition isn't satisfied here, so it's OK that $\mathcal{H} \neq E$. Energy is also not conserved in this system, since whoever is pulling up on the string is doing work on the system.

5.
A particle of mass m and velocity $\mathbf{v}_1$ leaves a semi-infinite space $z < 0$, where the potential energy is a constant U_1, and enters the remaining space $z > 0$, where the potential is a constant U_2.
(**a**)
Use symmetry arguments to find two constants of the motion.
Solution:
The lagrangian for this problem is

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U(z) ,$$

where U(z) is the potential energy, which clearly depends only on z. Applying the Euler-Lagrange equation:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) &= \frac{\partial \mathcal{L}}{\partial x} = 0 \\ m\dot{x} &= \text{constant} \\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) &= \frac{\partial \mathcal{L}}{\partial y} = 0 \\ m\dot{y} &= \text{constant} \end{aligned}$$

(**b**)
Use these two constants to obtain the new velocity $\mathbf{v}_2$.
Solution:
By the constants of part (a), $v_{2x} = v_{1x}$ and $v_{2y} = v_{1y}$. Since the lagrangian does not depend explicitly on time, we know that the total energy is conserved as well:

$$\begin{aligned} &\tfrac{1}{2}m{v_1}^2 + U_1 = \tfrac{1}{2}m{v_2}^2 + U_2 \\ &\frac{m}{2}\left({v_{1x}}^2 + {v_{1y}}^2 + {v_{1z}}^2\right) + U_1 \\ &\qquad = \frac{m}{2}\left({v_{2x}}^2 + {v_{2y}}^2 + {v_{2z}}^2\right) + U_2 \\ &\frac{m}{2}{v_{1z}}^2 + U_1 = \frac{m}{2}{v_{2z}}^2 + U_2 \\ &\qquad v_{2z} = \sqrt{{v_{1z}}^2 + \frac{2}{m}\left(U_1 - U_2\right)} \end{aligned}$$

We take the positive root, for if v_{2z} were negative, then the particle would not travel into the positive z space at all. Hence:

$$\mathbf{v}_2 = v_{1x}\,\hat{i} + v_{1y}\,\hat{j} + \sqrt{{v_{1z}}^2 + \frac{2}{m}\left(U_1 - U_2\right)}\,\hat{k}$$

6.
The Lagrangian for a (physically interesting) system is

$$\begin{aligned} &\mathcal{L}(\varphi, \dot{\varphi}, \theta, \dot{\theta}, \psi, \dot{\psi}, t) = \tfrac{1}{2}I(\dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2) \\ &+ \tfrac{1}{2}I_3(\dot{\varphi}\cos\theta + \dot{\psi})^2 - mgh\cos\theta , \end{aligned}$$

where (φ, θ, ψ) are Euler angles and (I, I_3, mgh) are constants.
(**a**)
Find two cyclic coordinates and obtain the two corresponding conserved canonically conjugate momenta.
Solution:
$\mathcal{L}$ is independent of φ and ψ, so they are cyclic.

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} &= I\dot{\varphi}\sin^2\theta + I_3\left(\dot{\varphi}\cos\theta + \dot{\psi}\right)\cos\theta \\ &= p_\varphi \quad \text{(a constant)} \\ \frac{\partial \mathcal{L}}{\partial \dot{\psi}} &= I_3\left(\dot{\varphi}\cos\theta + \dot{\psi}\right) \\ &= p_\psi \quad \text{(a constant)} \end{aligned}$$

(**b**)
Find a third constant of the motion.
Solution:
Since $\mathcal{L}$ does not depend explicitly on time, the Hamiltonian is equal to the total energy, and is thus conserved:

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\dot{\varphi} + \frac{\partial \mathcal{L}}{\partial \dot{\psi}}\dot{\psi} + \frac{\partial \mathcal{L}}{\partial \dot{\theta}}\dot{\theta} - \mathcal{L}$$

Inserting $\mathcal{L}$ and the other expressions from above, and simplifying, yields:

$$\begin{aligned} \mathcal{H} &= \tfrac{1}{2}I\left(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2\right) + \\ &\qquad \tfrac{1}{2}I_3\left(\dot{\varphi}\cos\theta + \dot{\psi}\right)^2 + mgh\cos\theta \\ &= \text{constant} \end{aligned}$$

(**c**)
Using the results of (**a**) and (**b**), express $\dot{\theta}^2$ as a function only of θ and constants.
Solution:
From our expressions in (a) we find that

$$\dot{\varphi} = \frac{p_\varphi - p_\psi \cos\theta}{I \sin^2\theta}$$
$$\dot{\psi} = \frac{p_\psi}{I_3} - \frac{(p_\varphi - p_\psi \cos\theta)\cos\theta}{I\sin^2\theta}$$

Substituting this into our expression for $\mathcal{H}$ yields:

$$\mathcal{H} = \tfrac{1}{2} I \left(\frac{(p_\varphi - p_\psi \cos\theta)^2}{I^2 \sin^2\theta} + \dot{\theta}^2 \right) + \tfrac{1}{2} I_3 \left(\frac{p_\psi}{I_3} \right)^2 + mgh\cos\theta$$

Solve for $\dot{\theta}^2$:

$$\dot{\theta}^2 = \frac{2\mathcal{H}}{I} - \frac{p_\psi^2}{I I_3} - \frac{2mgh\cos\theta}{I} - \frac{(p_\varphi - p_\psi\cos\theta)^2}{I^2\sin^2\theta}$$

7.
The interaction Lagrangian for a system consisting of a relativistic test particle of mass m and charge e moving in a static electromagnetic field is

$$\mathcal{L}(\mathbf{x}, \mathbf{v}, t) = -mc^2\sqrt{1 - \frac{v^2}{c^2}} + e\mathbf{v}\cdot\mathbf{A} - e\phi\ ,$$

where $\mathbf{x}$ is the particle's position, $\mathbf{v}$ is its velocity, $\phi(\mathbf{x})$ is the electrostatic potential ($\mathbf{E} = -\nabla\phi$), $\mathbf{A}$ is the (static) magnetic vector potential ($\mathbf{B} = \nabla \times \mathbf{A}$), and c is the speed of light.

(**a**)
Write down the canonical momenta (p_1, p_2, p_3) which are conjugate to the Cartesian coordinates (x_1, x_2, x_3).
Solution:

$$p_i = \frac{\partial \mathcal{L}}{\partial v_i} = \frac{mv_i}{\sqrt{1 - \frac{v^2}{c^2}}} + eA_i$$

(**b**)
Compute the Hamiltonian $\mathcal{H}(\mathbf{x}, \mathbf{v}, t)$.
Solution:

$$\begin{aligned}\mathcal{H} &= p_i v_i - \mathcal{L} \\ &= \frac{mv^2}{\sqrt{1 - \frac{v^2}{c^2}}} + e\mathbf{v}\cdot\mathbf{A} \\ &\quad - \left(-mc^2\sqrt{1 - \frac{v^2}{c^2}} + e\mathbf{v}\cdot\mathbf{A} - e\phi \right) \\ &= \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} + e\phi\end{aligned}$$

(**c**)
Re-express $\mathcal{H}(\mathbf{x}, \mathbf{v}, t)$ as the function $\mathcal{H}(\mathbf{x}, \mathbf{p}, t)$.
Solution:
From part (a), we can solve for the v's in terms of the p's:

$$v_i = \frac{c(p_i - eA_i)}{\sqrt{m^2c^2 + (\mathbf{p} - e\mathbf{A})^2}}$$

Inserting this into our answer to (b):

$$\begin{aligned}\mathcal{H} &= \frac{mc^2}{\sqrt{1 - \frac{1}{2}\left(c^2 \frac{(\mathbf{p} - e\mathbf{A})^2}{m^2c^2 + (\mathbf{p} - e\mathbf{A})^2} \right)}} + e\phi \\ &= \sqrt{m^2c^4 + (\mathbf{p} - e\mathbf{A})^2 c^2} + e\phi\end{aligned}$$

(**d**)
Show that $\mathcal{H}$ is conserved. Is it equal to

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}\ ,$$

the total (relativistic) energy of the test particle? Explain.
Solution:
Since $\mathcal{L}$ does not depend explicitly on t, $\frac{d\mathcal{H}}{dt} = 0$, and so $\mathcal{H}$ is conserved. However, $\mathcal{H}$ is *not* equal to the above expression for relativistic energy. This is because the above expression is derived assuming no external fields, such as the electric field generated by the ϕ which appears in our $\mathcal{H}$.

8.
Consider f and g to be any two continuous functions of the generalized coordinates q_i and canonically conjugate momenta p_i, as well as time:

$$f = f(q_i, p_i, t)$$
$$g = g(q_i, p_i, t)\ .$$

The *Poisson bracket* of f and g is defined by

$$[f, g] \equiv \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\ ,$$

where summation over i is implied. Prove the following properties of the Poisson bracket:
(**a**)

$$\frac{df}{dt} = [f, \mathcal{H}] + \frac{\partial f}{\partial t}$$

Solution:

$$[f, H] = \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} = \frac{\partial f}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial f}{\partial p_i}\frac{dp_i}{dt}$$

where the last step comes from Hamilton's equations of motion. By the chain rule,

$$\frac{df}{dt} = \frac{\partial f}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial f}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial f}{\partial t}$$

Compare these last two equations, and you see that $df/dt = [f, H] + \partial f/\partial t$.

(**b**)

$$\dot{q}_i = [q_i, \mathcal{H}]$$

Solution:

$$[q_i, H] = \frac{\partial q_i}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial H}{\partial q_k}$$

But $\partial q_i/\partial p_k = 0$ (p's and q's are independent variables), and $\partial q_i/\partial q_k = \delta_{ik}$. So only one term in the sum over k is nonzero:

$$[q_i, H] = \delta_{ik}\frac{\partial H}{\partial p_k} = \frac{\partial H}{\partial p_i} = \dot{q}_i$$

(by Hamilton's equations again.)

(**c**)

$$\dot{p}_i = [p_i, \mathcal{H}]$$

Solution:
Just like part (b): $[p_i, H] = \frac{\partial p_i}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial p_i}{\partial p_k}\frac{\partial H}{\partial q_k} = -\delta_{ik}\frac{\partial H}{\partial q_k} = -\frac{\partial H}{\partial q_i} = \dot{p}_i$.

(**d**)

$$[p_i, p_j] = 0$$

Solution:
$[p_i, p_j] = \frac{\partial p_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k}\frac{\partial p_j}{\partial q_k}$. But each term in this expression contains a $\partial p/\partial q$, and that's zero. So the whole expression is zero.

(**e**)

$$[q_i, q_j] = 0$$

Solution:
Same as part (d): Each term in the Poisson bracket contains a $\partial q/\partial p$, which is zero.

(**f**)

$$[q_i, p_j] = \delta_{ij}\ ,$$

where $\mathcal{H}$ is the Hamiltonian. If the Poisson bracket of two quantities is equal to unity, the quantities are said to be *canonically conjugate*. On the other hand, if the Poisson bracket vanishes, the quantities are said to *commute*.
Solution:

$$[q_i, p_j] = \frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_j}{\partial q_k} = \delta_{ik}\delta_{jk} - 0\cdot 0 = \delta_{ij}$$

(**g**)
Show that any quantity that does not depend explicitly on the time and that commutes with the Hamiltonian is a constant of the motion.
Solution:
"Constant of the motion" means $df/dt = 0$, so what we need to show is that if $[f, H] = \partial f/\partial t = 0$, then $df/dt = 0$. But that follows immediately from part (a), by simply substituting for $[f, H]$ and $\partial f/\partial t$.

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

ASSIGNMENT 5

Reading:
105 Notes 3.4-3.7
Hand & Finch 3.4-3.9

1.
A particle of mass m and electric charge q is situated in an alternating electric field directed along the x axis: $E_x = E_0 \cos\omega t$. The particle also experiences a force in the x direction proportional to the *third derivative* of its x coordinate:

$$F_\alpha = +\alpha \frac{d^3x}{dt^3} \ ,$$

where α is a positive constant. [This model gives an approximate description of a charged particle that scatters radiation.]

Find the amplitude and phase of the particle's oscillation in the steady state.

2.
Consider an extremely underdamped oscillator ($\omega_0/\gamma \equiv Q \gg 1$).
(**a**)
Suppose that the oscillator is undriven, but initially it is excited. How many oscillation periods are required for the energy stored in the oscillator to diminish by a factor of e?
(**b**)
Instead suppose that the oscillator is driven at resonance. What is the ratio of the energy stored in the oscillator to the work done by the driving force in one oscillation period?

3.
Consider an undriven oscillator satisfying the initial conditions $x(0) = x_0$, $\dot{x}(0) = 0$. Find $x(t)$ when the oscillator is...
(**a**)
...slightly underdamped ($\omega_0/\gamma \equiv Q = \frac{1}{\sqrt{2}}$).
(**b**)
...slightly overdamped ($\omega_0/\gamma \equiv Q = \frac{6}{13}$).

4.
Consider a critically damped oscillator ($\omega_0/\gamma \equiv Q = \frac{1}{2}$) that remains at rest at $x = 0$ for $t < 0$, but is driven at resonance by a force F_x such that

$$\frac{F_x}{m} = G \sin\omega_0 t$$

for $t > 0$, where G is a constant. Find $x(t)$.
[*Hint:* It is somewhat easier to solve this problem directly (matching boundary conditions at $t = 0$) than to use a Green function.]

5.
Woofer design. With compact discs well established as a recording medium, loudspeaker distortion is the last major barrier to true sound reproduction. A woofer in a sealed box ("acoustic suspension") is the simplest type to analyze. The motion of the cone of mass m is governed by the equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \cos\omega t \ ,$$

where F_0 is constant if the amplifier output resistance or the voice coil resistance is excessive (not a typical assumption, but one we will make here for simplicity). The average sound intensity is proportional to the average (acceleration)^2 of the cone. The damping factor b is proportional to the strength of the magnetic "motor" – the magnet and voice coil assembly. The spring constant k is inversely proportional to the volume of air sealed in the box.
(**a**)
Try to think up a simple mechanical test that you can perform in the showroom (when the salesperson is looking the other way) to see whether

the cone is underdamped or overdamped.
(**b**)
Suppose the assembly goes through resonance at $\nu_0 = 50$ Hz with $Q = 1$. (These are typical specifications for a medium quality classical music speaker.) By what factor will the sound intensity vary at 25 Hz? 100 Hz?
(**c**)
Sketch the effect upon smoothness of bass response of greatly increasing the cone area (to an inexperienced buyer, this often increases the speaker's apparent value). [*Hint*: Make reasonable assumptions concerning the dependence of m and k on the cone area.]

6.
Obtain the Fourier series that represents the function

$$\begin{aligned} F(t) &= 0 \;\; (-\frac{2\pi}{\omega} < t < 0) \\ &= F_0 \sin \omega t \;\; (0 < t < \frac{2\pi}{\omega}) \, . \end{aligned}$$

7.
Consider a damped oscillator (as usual, characterized by γ, ω_0, and m) driven by a periodic force F. During one period,

$$-\frac{2\pi}{\omega} < t < \frac{2\pi}{\omega} \, ,$$

F is taken to be equal to $F(t)$ in problem (**6.**); before and afterward, it simply repeats itself.

Find $x(t)$ for this oscillator. You may assume that any transient effects, due to the driving force having been turned on at $t = -\infty$, have damped out.

8.
Derive the Green function for an *overdamped* oscillator initially at rest at the origin. [*Hint*: Use the method of 105 Notes sections 3.5 and 3.6.]

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO PROBLEM SET 5
Solutions by T. Bunn and M. Strovink

Reading:
105 Notes 3.4-3.7
Hand & Finch 3.4-3.9

1.
A particle of mass m and electric charge q is situated in an alternating electric field directed along the x axis: $E_x = E_0 \cos\omega t$. The particle also experiences a force in the x direction proportional to the *third derivative* of its x coordinate:

$$F_\alpha = +\alpha \frac{d^3x}{dt^3} \ ,$$

where α is a positive constant. [This model gives an approximate description of a charged particle that scatters radiation.]

Find the amplitude and phase of the particle's oscillation in the steady state.
Solution:
The particle feels two forces: $F_{\rm elec} = qE = qE_0 \cos\omega t$, and $F_\alpha = \alpha\dddot{x}$. So its equation of motion is

$$m\ddot{x} = \alpha\dddot{x} + qE_0\, e^{i\omega t}$$

(We can replace $\cos\omega t$ by $e^{i\omega t}$ because their real parts are the same.) As usual, we solve this kind of equation by guessing that the answer is of the form $x = \mathrm{Re}\left(Ae^{i\omega t}\right)$. Then the equation of motion becomes

$$-mA\omega^2 = qE_0 - i\alpha A\omega^3$$

Solve this equation for A:

$$A = \frac{qE_0}{-m\omega^2 + i\alpha\omega^3} = -\frac{qE_0}{\omega^2}\frac{m + i\alpha\omega}{m^2 + \alpha^2\omega^2}$$

So A is a complex number of the form $A = |A|\, e^{i\varphi}$, with amplitude and phase

$$|A| = \frac{qE_0}{\omega^2\sqrt{m^2 + \alpha^2\omega^2}} \ , \ \ \varphi = \pi + \arctan\Big(\frac{\alpha\omega}{m}\Big) \ .$$

2.
Consider an extremely underdamped oscillator $(\omega_0/\gamma \equiv Q \gg 1)$.
(**a**)
Suppose that the oscillator is undriven, but initially it is excited. How many oscillation periods are required for the energy stored in the oscillator to diminish by a factor of e?
Solution:
The general solution to the undriven underdamped oscillator is

$$\begin{aligned} x(t) &= Be^{-\gamma t/2}\cos\left(\omega_\gamma t + \beta\right) \ , \ \text{where} \\ \omega_\gamma^2 &\equiv \omega_0^2 - \frac{\gamma^2}{4} \\ &= \omega_0^2\Big(1 - \frac{1}{4Q^2}\Big) \\ &\to \omega_0^2 \ \text{ for } Q \gg 1 \ . \end{aligned}$$

The total energy in the oscillator is the sum of kinetic and potential energy terms:

$$\begin{aligned} E &= \tfrac{1}{2}kx^2 + \tfrac{1}{2}m\dot{x}^2 \\ &= \tfrac{1}{2}m\omega_0^2 x_{\rm max}^2 \ , \end{aligned}$$

where as usual $\omega_0^2 \equiv k/m$, and we have used the fact that $\dot{x} = 0$ when x is at its maximum displacement $x_{\rm max}$. From the general solution, $x_{\rm max} \propto \exp\left(-\gamma t/2\right)$, so $E \propto \exp\left(-\gamma t\right)$. Therefore the time τ required for E to diminish by a factor e is $\tau = 1/\gamma$. In this time, the number N of periods is

$$N = \frac{\tau}{2\pi/\omega_\gamma} \approx \frac{\gamma^{-1}}{2\pi/\omega_0} = \frac{Q}{2\pi} \ .$$

(**b**)
Instead suppose that the oscillator is driven at

resonance. What is the ratio of the energy stored in the oscillator to the work done by the driving force in one oscillation period?
Solution:
As usual substitute $x \equiv \mathrm{Re}\big(Ae^{i\omega t}\big)$, and choose to solve the complex equation. When the driving force is $F_0 \cos \omega t$, this yields

$$-\omega^2 A + i\gamma\omega A + \omega_0^2 A = F_0/m \ .$$

At resonance ($\omega \equiv \omega_0$),

$$A = \frac{-iF_0}{\gamma\omega_0 m} \ .$$

Using the arguments in part (**a**), the energy in the oscillator is

$$\begin{aligned} E &= \tfrac{1}{2} m\omega_0^2 x_{\max}^2 \\ &= \tfrac{1}{2} m\omega_0^2 \frac{F_0^2}{\gamma^2\omega_0^2 m^2} \\ &= \frac{F_0^2}{2\gamma^2 m} \ . \end{aligned}$$

To determine the energy dissipated in one period, we integrate the work W done by the driving force:

$$\begin{aligned} W &= \oint F_x \, dx \\ &= \oint F_x \, \frac{dx}{dt} dt \\ &= \oint F_x v_x \, dt \ . \end{aligned}$$

Using our determination of A, we know $x(t)$ and therefore $v(t)$:

$$\begin{aligned} x(t) &= \mathrm{Re}\Big(\frac{-iF_0}{\gamma\omega_0 m} e^{i\omega_0 t}\Big) \\ &= \frac{F_0}{\gamma\omega_0 m} \sin \omega_0 t \\ v_x(t) &= \frac{F_0}{\gamma m} \cos \omega_0 t \ . \end{aligned}$$

Plugging $v_x(t)$ into the integral, and recalling that the square of any circular function has an average value of $\frac{1}{2}$ over one period T,

$$\begin{aligned} W = \oint F_x v_x \, dt &= \oint dt \, F_0 \cos \omega_0 t \, \frac{F_0}{\gamma m} \cos \omega_0 t \\ &= \frac{T}{2} \frac{F_0^2}{\gamma m} \\ &= \frac{2\pi}{2\omega_0} \frac{F_0^2}{\gamma m} \\ &= \frac{\pi}{Q} \frac{F_0^2}{\gamma^2 m} \ . \end{aligned}$$

Comparing W to E,

$$\frac{E}{W} = \frac{Q}{2\pi} \ ,$$

the same ratio obtained in part (**a**).

3.
Consider an undriven oscillator satisfying the initial conditions $x(0) = x_0$, $\dot{x}(0) = 0$. Find $x(t)$ when the oscillator is...
(**a**)
...slightly underdamped ($\omega_0/\gamma \equiv Q = \frac{1}{\sqrt{2}}$).
Solution:
The general solution to the undriven underdamped oscillator is

$$\begin{aligned} x(t) &= Be^{-\gamma t/2} \cos\left(\omega_\gamma t + \beta\right) \ , \text{ where} \\ \omega_\gamma^2 &\equiv \omega_0^2 - \frac{\gamma^2}{4} \\ &= \omega_0^2 \Big(1 - \frac{1}{4Q^2}\Big) \\ &= \frac{\omega_0^2}{2} \ \text{ when } \ Q = \frac{1}{\sqrt{2}} \\ x(t) &= Be^{-\omega_0 t/\sqrt{2}} \cos\big(\frac{\omega_0 t}{\sqrt{2}} + \beta\big) \ . \end{aligned}$$

Applying the boundary condition that the initial velocity vanishes,

$$\begin{aligned} 0 = \dot{x}(0) &= -\frac{\omega_0}{\sqrt{2}} B \cos \beta - B \frac{\omega_0}{\sqrt{2}} \sin \beta \\ \Rightarrow \cos \beta &= -\sin \beta \\ \beta &= -\frac{\pi}{4} \ . \end{aligned}$$

Finally, setting the initial displacement equal to x_0,

$$\begin{aligned} x_0 = x(0) &= B \cos \beta = B \cos\big(-\frac{\pi}{4}\big) \\ B &= \sqrt{2} x_0 \ . \end{aligned}$$

Putting it all together,

$$x(t) = \sqrt{2}x_0 e^{-\omega_0 t/\sqrt{2}} \cos\left(\frac{\omega_0 t}{\sqrt{2}} - \frac{\pi}{4}\right) .$$

(**b**)
...slightly overdamped ($\omega_0/\gamma \equiv Q = \frac{6}{13}$).
Solution:
The general solution to the undriven overdamped oscillator is

$$\begin{aligned} x(t) &= C_+ e^{-\gamma_+ t} + C_- e^{-\gamma_- t} \text{ , where} \\ \gamma_\pm &\equiv \frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \\ Q &= \tfrac{6}{13} \\ \Rightarrow \gamma &= \tfrac{13}{6}\omega_0 \\ \gamma_\pm &= \omega_0\left(\tfrac{13}{12} \pm \sqrt{(\tfrac{13}{12})^2 - 1}\right) \\ &= \omega_0\left(\tfrac{13}{12} \pm \tfrac{5}{12}\right) \\ \gamma_+ &= \tfrac{3}{2}\omega_0 \\ \gamma_- &= \tfrac{2}{3}\omega_0 \ . \end{aligned}$$

Applying the boundary condition that the initial velocity vanishes,

$$\begin{aligned} 0 = \dot{x}(0) &= -\gamma_+ C_+ - \gamma_- C_- \\ &= -\tfrac{3}{2}\omega_0 C_+ - \tfrac{2}{3}\omega_0 C_- \\ C_+ &= -\tfrac{4}{9} C_- \ . \end{aligned}$$

Finally, setting the initial displacement equal to x_0,

$$\begin{aligned} x_0 = x(0) &= C_+ + C_- \\ &= \left(-\tfrac{4}{9} + 1\right) C_- \\ C_- &= \tfrac{9}{5} x_0 \\ C_+ &= -\tfrac{4}{5} x_0 \ . \end{aligned}$$

Putting it all together,

$$x(t) = \frac{x_0}{5}\Big(9 \exp\left(-\tfrac{2}{3}\omega_0 t\right) - 4 \exp\left(-\tfrac{3}{2}\omega_0 t\right)\Big) \ .$$

4.
Consider a critically damped oscillator ($\omega_0/\gamma \equiv Q = \frac{1}{2}$) that remains at rest at $x = 0$ for $t < 0$, but is driven at resonance by a force F_x such that

$$\frac{F_x}{m} = G \sin \omega_0 t$$

for $t > 0$, where G is a constant. Find $x(t)$. [*Hint:* It is somewhat easier to solve this problem directly (matching boundary conditions at $t = 0$) than to use a Green function.]
Solution:
For time $t > 0$ the equation we're solving is of the form

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = G \sin \omega t,$$

which we've solved before. The particular solution is most easily written in complex notation as $x_p(t) = \text{Re}\left(Ae^{i\omega t}\right)$, where

$$A = \frac{-iG}{(\omega_0^2 - \omega^2) + i\gamma\omega} \ .$$

Substituting $\gamma = 2\omega_0$ (critical damping) and $\omega = \omega_0$ (driven at resonance) yields

$$A = -\frac{G}{\gamma\omega_0} = -\frac{G}{2\omega_0^2} \ .$$

We need to add a homogeneous solution of the form $x_h = D_1 e^{-\omega_0 t} + D_2 t e^{-\omega_0 t}$ to this to make it meet the initial conditions. The conditions are

$$\begin{array}{lll} x(0) = 0, & \text{so} & A + D_1 = 0 \\ \dot{x}(0) = 0, & \text{so} & -\omega_0 D_1 + D_2 = 0 \end{array}$$

Therefore

$$\begin{aligned} D_1 &= \frac{G}{2\omega_0^2} \\ D_2 &= \frac{G}{2\omega_0} \ . \end{aligned}$$

Putting it all together,

$$x(t) = \frac{G}{2\omega_0^2}\Big(e^{-\omega_0 t} + \omega_0 t e^{-\omega_0 t} - \cos \omega_0 t\Big) \ .$$

5.
Woofer design. With compact discs well established as a recording medium, loudspeaker distortion is the last major barrier to true sound reproduction. A woofer in a sealed box ("acoustic suspension") is the simplest type to analyze.

The motion of the cone of mass m is governed by the equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \cos\omega t \ ,$$

where F_0 is constant if the amplifier output resistance or the voice coil resistance is excessive (not a typical assumption, but one we will make here for simplicity). The average sound intensity is proportional to the average (acceleration)^2 of the cone. The damping factor b is proportional to the strength of the magnetic "motor" – the magnet and voice coil assembly. The spring constant k is inversely proportional to the volume of air sealed in the box.

(**a**)
Try to think up a simple mechanical test that you can perform in the showroom (when the salesperson is looking the other way) to see whether the cone is underdamped or overdamped.

Solution:
Just do something to displace the cone from its equilibrium position (by pushing on it or something). If it is overdamped, it will return straight to where it started. If it is underdamped, it will oscillate first.

(**b**)
Suppose the assembly goes through resonance at $\nu_0 = 50$ Hz with $Q = 1$. (These are typical specifications for a medium quality classical music speaker.) By what factor will the sound intensity vary at 25 Hz? 100 Hz?

Solution:
Remember that the resonant angular frequency $\omega_0 = \sqrt{k/m}$. How is Q defined? If you weren't given both γ, the damping coefficient, and ω_0, you could measure Q from $Q = \omega_0/\Delta\omega$, where $\Delta\omega$ is the full width at half maximum of the function $\omega^2 |A|^2$ (whose maximum is at ω_0). In our case, we do know $\gamma \equiv b/m$ and ω_0, so it is simpler to use the definition $Q = \omega_0/\gamma$. (If you weren't given γ you could measure it from the solution to the undriven oscillator: $x(t) = e^{-\gamma t/2}\cos\omega t$.) In our case, $Q = 1$, so $\gamma = \omega_0$.

Now, the equation of motion for our speaker is

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos\omega t \ .$$

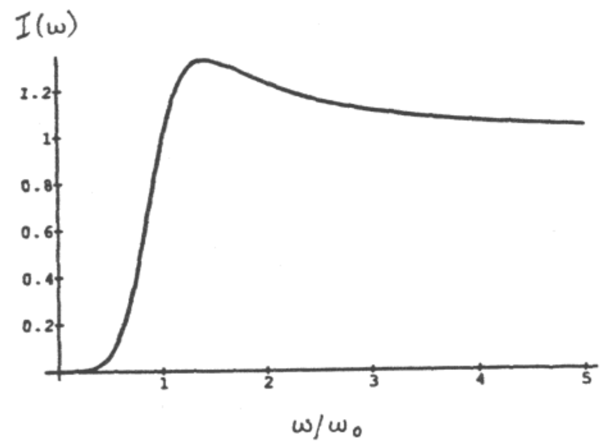
The steady-state solution must be of the form $x(t) = \mathrm{Re}(Ae^{i\omega t})$, with

$$\left(-m\omega^2 + ib\omega + k\right) A = F_0$$
$$A = \frac{F_0}{(k - m\omega^2) + ib\omega} = \frac{F_0/m}{(\omega^2 - \omega_0^2) + i\gamma\omega}$$

The sound intensity is proportional to the average acceleration squared, which is proportional to $\omega^4 |A|^2$. That is,

$$I(\omega) = \frac{C\omega^4}{\left(\omega^2 - \omega_0^2\right)^2 + \gamma^2\omega^2} = \frac{C\omega^4}{\left(\omega^2 - \omega_0^2\right)^2 + \omega_0^2\omega^2} \ ,$$

where C is a constant. This function is plotted below for your amusement.



The relevant facts about it for this problem, though, are that

$$\frac{I(\frac{1}{2}\omega_0)}{I(\omega_0)} = \frac{1}{13} = 0.0769$$
$$\frac{I(2\omega_0)}{I(\omega_0)} = \frac{16}{13} = 1.231 \ .$$

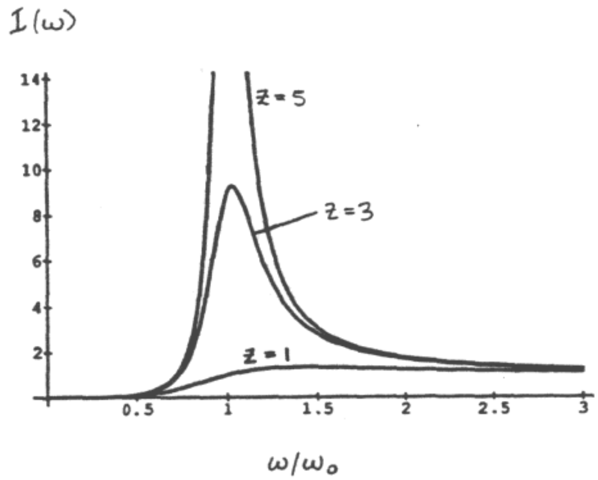
(**c**)
Sketch the effect upon smoothness of bass response of greatly increasing the cone area (to an inexperienced buyer, this often increases the speaker's apparent value). [*Hint*: Make reasonable assumptions concerning the dependence of m and k on the cone area.]

Solution:

What does increasing the cone size do to ω_0 and γ? Well, if $\mathcal{A}$ is the cone area, then $k \propto \mathcal{A}$ (because $k \propto F$, and force is pressure times area), and $m \propto \mathcal{A}$, so $\omega_0 = \sqrt{k/m}$ is independent of $\mathcal{A}$, and $\gamma = b/m \propto \mathcal{A}^{-1}$. So if we let $z = \mathcal{A}/\mathcal{A}_{\text{initial}}$ be the factor by which the area is expanded, then $\gamma = \omega_0/z$, and our expression for the sound intensity becomes

$$
\begin{aligned}
I(\omega) &= \frac{C\omega^4}{\left(\omega^2 - \omega_0^2\right)^2 + \gamma^2\omega^2} \\
&= \frac{C\omega^4}{\left(\omega^2 - \omega_0^2\right)^2 + \omega_0^2\omega^2/z^2} .
\end{aligned}
$$

Now we can plot $I(\omega)$ for various values of z to see how smooth the response is.



From the graph, we see that the audio response curve gets much less smooth *vs.* frequency as z increases. (The physical explanation for this is that the Q of the oscillator goes up as the cone size increases, so the peak is sharper.) You could come up with all kinds of ways to make this statement more precise. One would be to define the "smoothness" ratio R to be $I(2\omega_0)/I(\omega_0)$ (so that $R = 1$ is the desirable state), and plot R vs. z. By this method of measuring, you'd find that the smoothness indeed does get worse as the cone size increases.

6.

Obtain the Fourier series that represents the function

$$
\begin{aligned}
F(t) &= 0 \;\; (-\frac{2\pi}{\omega} < t < 0) \\
&= F_0 \sin\omega t \;\; (0 < t < \frac{2\pi}{\omega}) .
\end{aligned}
$$

Solution:

Remember: Any function $F(t)$ that repeats itself with period $T = 2\pi/\Omega$ and whose average value is zero can be expanded in a series

$$
\begin{aligned}
F(t) &= \sum_{n=1}^{\infty} \left(f_n \sin n\Omega t + g_n \cos n\Omega t\right) \;, \text{ where} \\
f_n &= \frac{\Omega}{\pi}\int_0^T F(t)\sin n\Omega t \\
g_n &= \frac{\Omega}{\pi}\int_0^T F(t)\cos n\Omega t \;.
\end{aligned}
$$

In our case, $F(t)$ repeats itself with period $T = 4\pi/\omega$, so $\Omega = \omega/2$. Then

$$
\begin{aligned}
f_n &= \frac{\omega}{2\pi}\int_{-2\pi/\omega}^{2\pi/\omega} F(t)\sin\tfrac{1}{2}n\omega t\,dt \\
&= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} F_0 \sin\omega t\,\sin\tfrac{1}{2}n\omega t\,dt \;.
\end{aligned}
$$

The best way to do integrals like these is to look them up, but if you're too proud, you can also calculate them by using the angle addition formulæ (in reverse) to write the integrand as a difference of two cosines. In either case, the answer comes out to be

$$
f_2 = \tfrac{1}{2}F_0 \qquad \text{and} \qquad f_n = 0 \quad \text{for } n \neq 2 \;.
$$

Now on to the g_n's: The relevant integral this time is

$$
\begin{aligned}
g_n &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} F_0 \sin\omega t\,\cos\tfrac{1}{2}n\omega t\,dt \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{\pi(4-n^2)}F_0 & \text{if } n \text{ is odd} \end{cases}
\end{aligned}
$$

So the answer is

$$
F(t) = \tfrac{1}{2}F_0\sin\omega t + \sum_{n\ \text{odd}} \frac{4}{\pi\left(4-n^2\right)}F_0\cos\tfrac{1}{2}n\omega t
$$

7.
Consider a damped oscillator (as usual, characterized by γ, ω_0, and m) driven by a periodic force F. During one period,

$$-\frac{2\pi}{\omega} < t < \frac{2\pi}{\omega} ,$$

F is taken to be equal to $F(t)$ in problem (**6.**); before and afterward, it simply repeats itself.

Find $x(t)$ for this oscillator. You may assume that any transient effects, due to the driving force having been turned on at $t = -\infty$, have damped out.

Solution:
Expressing $F(t)$ in terms of the Fourier series obtained in the previous problem, we need a particular solution to the differential equation

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{2m}\sin\omega t + \sum_{n\ \mathrm{odd}} \frac{4}{\pi(4-n^2)}\frac{F_0}{m}\cos\tfrac{n}{2}\omega t .$$

A linear oscillator must simultaneously respond at all frequencies at which it is driven, so we seek solutions of the form

$$x(t) = \mathrm{Re}\Big(Ae^{i\omega t} + \sum_{n\ \mathrm{odd}} B_n e^{i\frac{n}{2}\omega t}\Big) .$$

We substitute this form of x in the differential equation, and, as usual, we choose to solve the complex version of it. To do so, on the left *vs.* right-hand sides, we equate the coefficients of each factor of the form $\exp(i\Omega t)$, where Ω is equal either to ω or to $\frac{n}{2}\omega$ for any odd n. This yields the set of equations

$$A = \frac{-iF_0/2m}{(\omega_0^2-\omega^2)+i\gamma\omega} \qquad B_n = \frac{4}{\pi\,(4-n^2)}\frac{F_0/m}{\big(\omega_0^2-(\frac{n}{2})^2\omega^2\big)+i\gamma\frac{n}{2}\omega} .$$

(Note that A is of the same form as in problem (**4.**), and that the B_n are of standard form except for their n-dependent coefficients.) The solution is thus

$$x(t) = \mathrm{Re}\Big(\frac{-iF_0/2m}{(\omega_0^2-\omega^2)+i\gamma\omega}e^{i\omega t} + \sum_{n\ \mathrm{odd}} \frac{4}{\pi\,(4-n^2)}\frac{F_0/m}{(\omega_0^2-(\frac{n}{2})^2\omega^2)+i\gamma\frac{n}{2}\omega}e^{i\frac{n}{2}\omega t}\Big) .$$

If you are fond of such things, you can reexpress the contents of the large parentheses as a sum of purely real or imaginary parts, or rewrite $x(t)$ in terms of many different cosines and phases.

8.
Derive the Green function for an *overdamped* oscillator initially at rest at the origin. [*Hint*: Use the method of 105 Notes sections 3.5 and 3.6.]
Solution:
The equation for the Green function (call it X_g) is $\ddot{X}_g + \gamma\dot{X}_g + \omega_0^2 X_g = \delta(t)$. For $t > 0$ the right-hand side is zero, so we can just write down a solution to the homogeneous equation:

$$X_g = A_1\,e^{\alpha_+ t} + A_2\,e^{\alpha_- t} \ \text{ where}$$
$$\alpha_\pm = -\tfrac{1}{2}\gamma \pm \tfrac{1}{2}\sqrt{\gamma^2-4\omega_0^2} ,$$

and A_1 and A_2 are constants determined by the initial conditions. What are the initial conditions? Well, $X_g(0) = 0$, because the oscillator hasn't had any time to move immediately after the impulse, and $\dot{X}_g(0) = 1$, because the impulse per unit mass supplied by the δ-function is $\Delta v = \int \delta(t)\,dt = 1$. Solve these two equations and you get

$$A_1 = -A_2 = \frac{1}{\alpha_+ - \alpha_-} = \frac{1}{\sqrt{\gamma^2-4\omega_0^2}} .$$

So the Green function is

$$X_g(t) = \frac{1}{\sqrt{\gamma^2-4\omega_0^2}}\left(e^{\alpha_+ t} - e^{\alpha_- t}\right) .$$

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

ASSIGNMENT 6

Reading:
105 Notes 7.1-7.8
Hand & Finch 4.1-4.6

1.
Show that the period of oscillation of a particle of mass m in a potential $U = A|x|^n$ is given by

$$T = \frac{2}{n}\sqrt{\frac{2\pi m}{E}}\Big(\frac{E}{A}\Big)^{1/n}\frac{\Gamma\big(\frac{1}{n}\big)}{\Gamma\big(\frac{1}{2}+\frac{1}{n}\big)}$$

Take $n = 2$, evaluate the gamma functions, and thus show that T reduces to the normal expression for a parabolic potential.

2.
Use a Green function to obtain the response of an underdamped linear oscillator

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = F(t)$$

to a driving (acceleration) function of the form

$$F(t) = 0\ (t < 0);\ = F_0 e^{-\beta t}\ (t > 0)\ ,$$

where γ, ω_0, F_0, and β are constants.

3.
Consider a nonlinear damped oscillator whose motion is described by

$$\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt}\Big|\frac{dx}{dt}\Big| + \omega_0^2 x = 0$$

The initial conditions are $x(0) = a,\ \dot{x}(0) = 0$. Use the method of perturbations to find a solution that is accurate to first order in the small quantity λ.

4.
Two particles moving under the influence of their mutual gravitational force describe circular orbits about one another with period τ. If they are suddenly stopped in their orbits and allowed to gravitate toward each other, show that they will collide after a time $\tau/(4\sqrt{2})$.

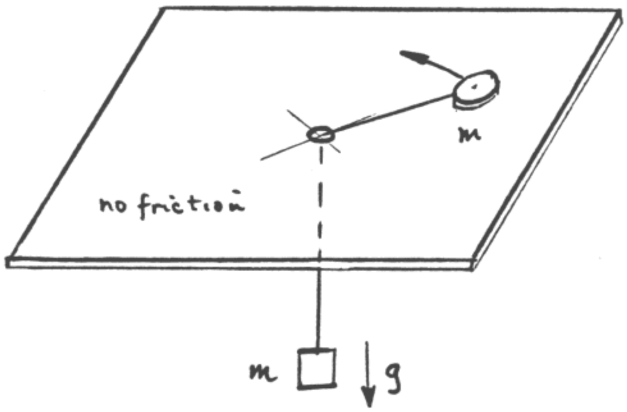
5.
A spacecraft in uniform circular orbit about the sun, far from any planet, consists of a nose cone and a service module. By means of explosive bolts, the nose cone separates from the service module. The direction of motion of the nose cone is unchanged, but its orbit becomes a parabola instead of a circle. The service module falls directly into the sun. Solve for the ratio $\rho = m_{\text{cone}}/m_{\text{spacecraft}}$, where the spacecraft mass is considered to be the sum of the nose cone and service module masses.

6.
A particle moves under the influence of a central force given by $F(r) = -k/r^n$. If the particle's orbit is circular and passes through the force center, show that $n = 5$.

7.
A spacecraft in circular orbit about the sun fires its thruster in order to change instantaneously the *direction* of its velocity $\mathbf{v}$ by 45° (toward the sun), keeping the same *magnitude* $|\mathbf{v}|$. What is the eccentricity of the spacecraft's new orbit?

8.

A puck of mass m is connected by a massless string to a weight of the same mass. It moves without friction on a horizontal table, in circular orbit about the hole.



(**a**)

Calculate the frequency of small radial oscillations about the circular orbit.

(**b**)

Expressing this frequency as a ratio to the orbital frequency, show that the orbit does not close.

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SOLUTION TO PROBLEM SET 6
Solutions by J. Barber and T. Bunn

Reading:
105 Notes 7.1-7.8
Hand & Finch 4.1-4.6

1.
Show that the period of oscillation of a particle of mass m in a potential $U = A|x|^n$ is given by

$$T = \frac{2}{n}\sqrt{\frac{2\pi m}{E}}\left(\frac{E}{A}\right)^{1/n}\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}$$

Take $n = 2$, evaluate the gamma functions, and thus show that T reduces to the normal expression for a parabolic potential.
Solution:
$E = \frac{1}{2}mv^2 + A\,|x|^n$, so

$$v = \dot{x} = \sqrt{\frac{2}{m}}\,(E - A\,|x|^n)^{\frac{1}{2}} \ .$$

We'll just compute the time it takes the particle to go from $x = 0$ to $x = x_{\max} \equiv (E/A)^{1/n}$. This time is one fourth of the total period T. Since x is positive over this interval, we can drop the absolute value signs.

$$\begin{aligned}\tfrac{1}{4}T &= \int dt = \int_0^{x_{\max}} \frac{dt}{dx}\,dx \\ &= \sqrt{\frac{m}{2}}\int_0^{x_{\max}} \frac{dx}{\sqrt{E - Ax^n}}\end{aligned}$$

Substitute $y = Ax^n/E$, and you get

$$T = \frac{2}{n}\sqrt{\frac{2m}{E}}\left(\frac{E}{A}\right)^{1/n}\int_0^1 \frac{y^{\frac{1}{n}-1}\,dy}{\sqrt{1-y}}$$

This integral is a beta function, which has the following properties:

$$B(p,q) \equiv \int_0^1 t^{p-1}(1-t)^{q-1}dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

In our case, $p = \frac{1}{n}$ and $q = \frac{1}{2}$. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, so

$$T = \frac{2}{n}\sqrt{\frac{2\pi m}{E}}\left(\frac{E}{A}\right)^{1/n}\frac{\Gamma(\frac{1}{n})}{\Gamma(\frac{1}{2}+\frac{1}{n})}.$$

2.
Use a Green function to obtain the response of an underdamped linear oscillator

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = F(t)$$

to a driving (acceleration) function of the form

$$\begin{aligned}F(t) &= 0 && (t < 0)\\ &= F_0 e^{-\beta t} && (t > 0)\ ,\end{aligned}$$

where γ, ω_0, F_0, and β are constants.
Solution:
The Green's function for an underdamped oscillator is

$$G(t) = \frac{1}{\omega_\gamma}e^{-\gamma t/2}\sin\omega_\gamma t.$$

So Green's method gives

$$\begin{aligned}x(t) &= \int_{-\infty}^{t} F(t')G(t-t')\,dt' \\ &= \frac{F_0}{\omega_\gamma}\int_0^t e^{-\gamma(t-t')/2}\sin\omega_\gamma(t-t')e^{-\beta t'}\,dt' \\ &= \frac{F_0 e^{-\beta t}}{\omega_\gamma}\int_0^t e^{(\beta-\gamma/2)u}\sin\omega_\gamma u\,du \quad (u \equiv t-t') \\ &= \frac{F_0}{\omega_\gamma\left((\beta-\gamma/2)^2+\omega_\gamma^2\right)} \\ &\times\Big[e^{-\gamma t/2}\left((\beta-\gamma/2)\sin\omega_\gamma t - \omega_\gamma\cos\omega_\gamma t\right) \\ &+ \omega_\gamma e^{-\beta t}\Big]\end{aligned}$$

3.
Consider a nonlinear damped oscillator whose motion is described by

$$\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt}\left|\frac{dx}{dt}\right| + \omega_0^2 x = 0$$

The initial conditions are $x(0) = a$, $\dot{x}(0) = 0$. Use the method of perturbations to find a solution that is accurate to first order in the small quantity λ.

Solution:
We write the solution $x(t)$ as a power series in λ, and assume that, since λ is small, we can drop all terms with two or more powers of λ. Then, setting $x(t) = x_0(t) + \lambda x_1(t)$, our equation of motion becomes

$$\ddot{x}_0 + \lambda\ddot{x}_1 + \lambda x_0\,|x_0| + \omega_0^2 x_0 + \lambda\omega_0^2 x_1 = 0$$

This equation must be true for all values of λ, so we really have two equations:

$$\ddot{x}_0 + \omega_0^2 x_0 = 0 \;\text{ and }\; \ddot{x}_1 + \omega_0^2 x_1 = -\dot{x}_0\,|\dot{x}_0|$$

The first equation is just the simple harmonic oscillator, and the solution which meets our initial conditions is $x_0(t) = a\cos\omega_0 t$. Substituting that into the second equation, we find that the equation for x_1 looks like a driven oscillator with a funny driving force:

$$\ddot{x}_1 + \omega_0^2 x_1 = a^2\omega_0^2 \sin\omega_0 t\,|\sin\omega_0 t|$$

The easiest way to solve this equation is to solve it first for the first half-period $0 < \omega_0 t < \pi$ and then for the second half-period. So during the first time interval,

$$\ddot{x}_1 + \omega_0^2 x_1 = a^2\omega_0^2\sin^2\omega_0 t = \tfrac{1}{2}a^2\omega_0^2(1 - \cos 2\omega_0 t).$$

If we guess that the particular solution is of the form $x_{1p} = A + B\cos 2\omega_0 t$, we can solve for A and B and get $x_{1p}(t) = \frac{1}{2}a^2\left(1 + \frac{1}{3}\cos 2\omega_0 t\right)$. We need to add a homogeneous solution to match the initial conditions $x_1(0) = 0$ and $\dot{x}_1(0) = 0$ (Remember: x_0 contained the initial displacement from the origin.) The homogeneous solution that does it is $x_{1h}(t) = -\frac{2}{3}a^2\cos\omega_0 t$, so the total solution is

$$x_1(t) = \frac{1}{2}a^2\left(1 + \frac{1}{3}\cos 2\omega_0 t - \frac{4}{3}\cos\omega_0 t\right)$$
$$(\text{for } 0 < \omega_0 t < \pi)\ .$$

For the second half-period, the driving force is $-a^2\omega_0^2\sin^2\omega_0 t$, exactly the negative of what it was before. So the particular solution x_{1p} will also be -1 times what it was before: $x_{1p}(t) = -\frac{1}{2}a^2\left(1 + \frac{1}{3}\cos 2\omega_0 t\right)$. But this time the initial conditions come from the fact that the position and velocity must be continuous at $t = \pi/\omega_0$. Specifically, from above equation for x_1, we have

$$x_1(\pi/\omega_0) = \tfrac{4}{3}a^2 \;\text{ and }\; \dot{x}_1(\pi/\omega_0) = 0$$

The homogeneous solution that matches these initial conditions is $x_{1h} = -2a^2\cos\omega_0 t$, so

$$x_1(t) = -a^2\left(\frac{1}{2} + \frac{1}{6}\cos 2\omega_0 t + 2\cos\omega_0 t\right)$$
$$(\text{for } \pi < \omega_0 t < 2\pi)\ .$$

In principle you'd need to repeat this process for each half-period *ad infinitum*, but we can see what's going to happen: After one full cycle the oscillator is again at rest, but it's at $x(T) = x_0(T) + \lambda x_1(T) = a(1 - \frac{8}{3}\lambda a)$, rather than at $x(T) = a$, which is where it would be if there were no damping. So it'll just repeat the same pattern as before, but with an amplitude smaller than before by this factor.

4.
Two particles moving under the influence of their mutual gravitational force describe circular orbits about one another with period τ. If they are suddenly stopped in their orbits and allowed to gravitate toward each other, show that they will collide after a time $\tau/(4\sqrt{2})$.

Solution:
It's much easier to work with the equivalent one-body problem, where r, the distance between the particles, is regarded as the distance to some fixed center of force, and the reduced mass μ takes the place of the mass. Then if R is the radius of the circular orbit, the period τ can be

found by setting the gravitational force equal to the centripetal force:

$$\frac{k}{R^2} = \frac{\mu v^2}{R} = \frac{4\pi^2 \mu R}{\tau^2}.$$

So $\tau = 2\pi\sqrt{\mu R^3/k}$.

Now let's find T, the time it takes the particles to collide, starting from rest at a distance R. By energy conservation, the speed at a distance r from the origin satisfies $\frac{1}{2}\mu v^2 - \frac{k}{r} = -\frac{k}{R}$, so

$$v = \dot{r} = -\sqrt{\frac{2k}{\mu}\left(\frac{1}{r} - \frac{1}{R}\right)},$$

where we take the negative root because r will be decreasing. Now we can get T by integrating:

$$\begin{aligned} T &= \int_R^0 \frac{dt}{dr}\,dr = \sqrt{\frac{\mu}{2k}} \int_0^R \left(\frac{1}{r} - \frac{1}{R}\right)^{-\frac{1}{2}} dr \\ &= \sqrt{\frac{\mu}{2k}} \int_0^R r^{\frac{1}{2}} \left(1 - \frac{r}{R}\right)^{-\frac{1}{2}} dr \\ &= \sqrt{\frac{\mu R^3}{2k}} \int_0^1 u^{\frac{1}{2}} (1-u)^{-\frac{1}{2}}\,du, \;\; \textit{where } u = \frac{r}{R} \end{aligned}$$

Using the definition of the Beta function from problem 1, with $p = \frac{3}{2}$ and $q = \frac{1}{2}$, we find that

$$\begin{aligned} T &= \sqrt{\frac{\mu R^3}{2k}} \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)} \\ &= \frac{\pi}{2}\sqrt{\frac{\mu R^3}{2k}} \\ &= \frac{\tau}{4\sqrt{2}} \end{aligned}$$

5.

A spacecraft in uniform circular orbit about the sun, far from any planet, consists of a nose cone and a service module. By means of explosive bolts, the nose cone separates from the service module. The direction of motion of the nose cone is unchanged, but its orbit becomes a parabola instead of a circle. The service module falls directly into the sun. Solve for the ratio $\rho = m_{\text{cone}}/m_{\text{spacecraft}}$, where the spacecraft mass is considered to be the sum of the nose cone and service module masses.

Solution:

Let v_0 be the initial velocity of the spacecraft (before the separation). Remember that an object in a parabolic orbit has twice as much kinetic energy as an identical object in a circular orbit at the same distance. (Why? Because an object in a circular orbit has $T = -\frac{1}{2}V$, by the virial theorem, and an object in a parabolic orbit has $T = -V$, since its total energy is zero.) So during the separation, the nose cone picks up a factor of two in kinetic energy, so its speed goes up by a factor $\sqrt{2}$: $v_{\text{cone}} = v_0\sqrt{2}$. Therefore, the change in momentum of the cone is $\Delta p_{\text{cone}} = m_{\text{cone}}\Delta v_{\text{cone}} = m_{\text{cone}} v_0(\sqrt{2} - 1)$.

Now consider the service module. Its speed is reduced to zero, so its change in momentum is $\Delta p_{\text{module}} = -m_{\text{module}} v_0$. The total change in momentum is zero, so

$$\begin{aligned} m_{\text{cone}} v_0(\sqrt{2} - 1) &= m_{\text{module}} v_0 \\ m_{\text{cone}}(\sqrt{2} - 1) &= m_{\text{module}} \\ \rho = \frac{m_{\text{cone}}}{m_{\text{cone}} + m_{\text{module}}} &= \frac{1}{\sqrt{2}} \end{aligned}$$

6.

A particle moves under the influence of a central force given by $F(r) = -k/r^n$. If the particle's orbit is circular and passes through the force center, show that $n = 5$.

Solution:

The equation in polar coordinates of a circle of radius a passing through the origin is

$$r = 2a\cos\theta$$

The other two facts we'll need are conservation of energy and angular momentum. Angular momentum conservation gives

$$l = mr^2\dot{\theta}, \qquad \text{or} \qquad \dot{\theta} = \frac{l}{mr^2}$$

with l constant. We can use this to write $\dot{r}$ in terms of r. Differentiate and square the equation for the circle, and substitute for $\dot{\theta}$:

$$\begin{aligned} \dot{r}^2 &= 4a^2 \sin^2\theta\,\dot{\theta}^2 \\ &= 4a^2\dot{\theta}^2(1 - \cos^2\theta) = \dot{\theta}^2(4a^2 - r^2) \\ &= \frac{l^2}{m^2 r^4}(4a^2 - r^2)\,. \end{aligned}$$

You're probably wondering why we're playing these algebra games. Well, we wanted to write the total energy of the particle without any derivatives in it. Noting that the potential energy corresponding to this force is $U(r) = -\frac{k}{(n-1)r^{n-1}}$, we can now write:

$$\begin{aligned} E &= \tfrac{1}{2}mr^2\dot{\theta}^2 + \tfrac{1}{2}m\dot{r}^2 - \frac{k}{(n-1)r^{n-1}} \\ &= \frac{l^2}{2mr^2} + \frac{l^2}{2mr^4}(4a^2 - r^2) - \frac{k}{(n-1)r^{n-1}} \\ &= \frac{2l^2a^2}{m}\frac{1}{r^4} - \frac{k}{(n-1)}\frac{1}{r^{n-1}} \end{aligned}$$

So the energy has two terms, one of which varies as r^{-4} and the other of which varies as $r^{-(n-1)}$. But E must be constant as r varies, so those two terms must cancel each other. That can only happen if their exponents are equal, so we must have $n = 5$.

7.

A spacecraft in circular orbit about the sun fires its thruster in order to change instantaneously the *direction* of its velocity $\mathbf{v}$ by 45° (toward the sun), keeping the same *magnitude* $|\mathbf{v}|$. What is the eccentricity of the spacecraft's new orbit?

Solution:

We shall use the subscript "f" to indicate immediately after the thruster fires, and "i" to indicate immediately before. We are given that

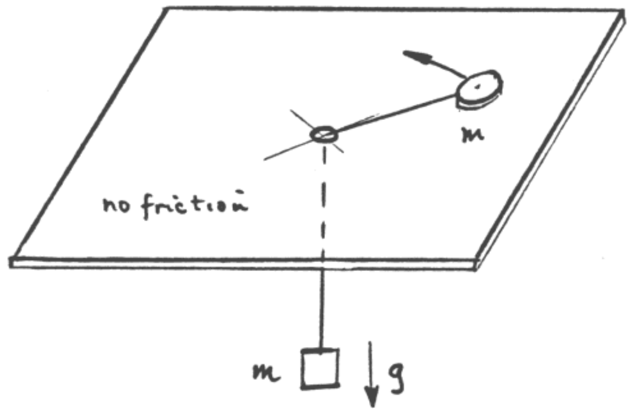
$$\vec{v}_\mathrm{f} = \frac{|\vec{v}_\mathrm{i}|}{\sqrt{2}}\left(-\hat{r} + \hat{\theta}\right) \, ,$$

from which we see that $l_\mathrm{f} = \frac{l_\mathrm{i}}{\sqrt{2}}$, where l is the angular momentum. Furthermore, $|\vec{v}_\mathrm{f}| = |\vec{v}_\mathrm{i}|$, and r remains instantaneously unchanged during the thrust, so $E_\mathrm{f} = E_\mathrm{i}$ as well. But $E = -\frac{k}{2a}$ (Notes 7.12), which implies that $a_\mathrm{f} = a_\mathrm{i}$. Using the definition of a from Notes 7.10 yields

$$\begin{aligned} a_\mathrm{i} &= a_\mathrm{f} \\ \frac{l_\mathrm{i}^2}{\mu k\left(1-\epsilon_\mathrm{i}^2\right)} &= \frac{l_\mathrm{f}^2}{\mu k\left(1-\epsilon_\mathrm{f}^2\right)} \\ \frac{l_\mathrm{i}^2}{\mu k} &= \frac{l_\mathrm{i}^2/2}{\mu k\left(1-\epsilon_\mathrm{f}^2\right)} \quad (\text{since } \epsilon_\mathrm{i} = 0) \\ 1 - \epsilon_\mathrm{f}^2 &= \tfrac{1}{2} \\ \epsilon_\mathrm{f} &= \frac{1}{\sqrt{2}} \end{aligned}$$

8.

A puck of mass m is connected by a massless string to a weight of the same mass. It moves without friction on a horizontal table, in circular orbit about the hole.



(a)

Calculate the frequency of small radial oscillations about the circular orbit.

Solution:

Let's take as our generalized coordinates r and θ, the polar coordinates of the puck with respect to the hole. Then

$$\begin{aligned} T &= T_\mathrm{puck} + T_\mathrm{weight} \\ &= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{m}{2}\dot{r}^2 \\ &= m\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2 \\ U &= mgr + \text{constant} \end{aligned}$$

and so the Lagrangian is:

$$\mathcal{L} = m\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2 - mgr$$

Since θ is cyclic, angular momentum is conserved, and $l = mr^2\dot{\theta}$. Applying the Euler-Lagrange equation to the r coordinate yields

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) &= \frac{\partial \mathcal{L}}{\partial r} = mr\dot{\theta}^2 - mg \\ 2\ddot{r} &= \frac{l^2}{m^2r^3} - g \end{aligned}$$

In a circular orbit, $r = R$ and $\ddot{r} = 0$, which yields $l^2 = m^2 R^3 g$ for a circular orbit. Now, suppose we start out in a circular orbit of radius R, but then perturb it by an amount x, where $x << R$, i.e. $r = R + x$. Using the value of l for a circular orbit, the D.E. then becomes:

$$\begin{aligned}
2\ddot{x} &= g\left(\frac{R^3}{(R+x)^3} - 1\right) \\
\ddot{x} &= \frac{g}{2}\left(\frac{1}{\left(1+\frac{x}{R}\right)^3} - 1\right) \\
\ddot{x} &\approx -\frac{3g}{2R}x \quad \text{(expand about } x = 0\text{)} \\
\omega_{\text{perturbation}} &= \sqrt{\frac{3g}{2R}}
\end{aligned}$$

(**b**)
Expressing this frequency as a ratio to the orbital frequency, show that the orbit does not close.
Solution:
From the expression for angular momentum in a circular orbit

$$\begin{aligned}
l^2 &= m^2 R^3 g = m^2 v^2 R^2 \\
v &= \sqrt{Rg}\ ,
\end{aligned}$$

and so the orbital frequency is

$$\omega_{\text{orbit}} = \frac{2\pi}{T} = 2\pi\frac{v}{2\pi R} = \sqrt{\frac{g}{R}}\ .$$

$$\frac{\omega_{\text{perturbation}}}{\omega_{\text{orbit}}} = \sqrt{\frac{3}{2}}$$

which is irrational, so the orbit does not close.

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

ASSIGNMENT 7

Reading:
105 Notes 8.1-8.3, 6.1-6.2 (again).
Hand & Finch 4.7, 5.1.

1.
[This a (hopefully clearer) version of Hand & Finch 4.17, "tetherball".] A mass m is attached to a weightless string that initially has a length s_0. The other end of the string is attached to a post of radius a. Neglect the effect of gravity. Suppose that the mass is set into motion. It is given an initial velocity of magnitude v_0 directed so that the string remains taut. The string wraps itself around the post, causing the mass to spiral inward toward it.

(**a**)
Write the Lagrangian in terms of $\dot{x}$ and $\dot{y}$, the cartesian velocity components of the mass. Is there a potential energy term?

(**b**)
Use as generalized coordinates $s(t)$, the length of the part of the string that is not yet in contact with the post, and $\psi(t)$, the azimuthal angle at which the string barely fails to make contact with the post. Express $\dot{x}$ and $\dot{y}$ in terms of these generalized coordinates and their time derivatives.

(**c**)
Write a (constraint) equation relating $\dot{s}$ to $\dot{\psi}$. Use it to greatly simplify your answers for (**b**). Rewrite the Lagrangian using s as the only generalized coordinate.

(**d**)
Use the Euler-Lagrange equation to obtain an equation of motion for s. (You don't need to solve it.)

(**e**)
Since the Lagrangian has no explicit time dependence, and it depends quadratically on $\dot{s}$, the total energy is conserved. Write an equation setting the initial energy (expressed in terms of v_0) equal to the energy at an arbitrary value of s (expressed in terms of s and $\dot{s}$).

(**f**)
Use this equation to express dt in terms of ds multiplied by a function of s. Integrate it to solve for the time T that elapses before the mass hits the post. You should obtain the simple result

$$T = \frac{s_0^2}{2av_0} \ .$$

(**g**)
Is the angular momentum of the mass about the *axis* of the post conserved in this problem? Why or why not?

2.
Hand & Finch 4.19.

3.
Hand & Finch 4.21.

4.
Consider a particle of mass m that is constrained to move on the surface of a paraboloid whose equation (in cylindical coordinates) is $r^2 = 4az$. If the particle is subject to a gravitational force $-mg\hat{\mathbf{z}}$, show that the frequency of small oscillations about a circular orbit with radius $\rho = \sqrt{4az_0}$ is

$$\omega = \sqrt{\frac{2g}{a + z_0.}}$$

5.
An orbit that is almost circular can be considered to be a circular orbit to which a small perturbation has been applied. Take ρ to be the (unperturbed) circular orbit radius and define

$$g(r) = \frac{1}{\mu}\frac{\partial U(r)}{\partial r} \ ,$$

where μ is the reduced mass and U is an arbitrary potential. Set the radius $r = \rho + x$, where x is a small perturbation.

(a)

Starting from the differential equation for r and using the fact that the angular momentum l is constant, substitute $r = \rho + x$. Retaining terms only to first order in x, Taylor expand $g(r)$ about the point $r = \rho$, and show that x satisfies the differential equation

$$\ddot{x} + \left[\frac{3g(\rho)}{\rho} + g'(\rho)\right] x = 0 \ ,$$

where $g'(\rho)$ is dg/dr evaluated at $r = \rho$.

(b)

Taking the force law to be $F(r) = -kr^{-n}$, where n is an integer, show that the angle between two successive values of $r = r_{\max}$ (the "apsidal angle") is $2\pi/\sqrt{3-n}$. Thus, if $n > -6$, show that in general a closed orbit will result only for the harmonic oscillator force and the inverse square law force.

6.

Consider the motion of a particle in a central force field $F = -k/r^2 + C/r^3$.

(a)

Show that the equation of the orbit can be put in the form

$$\frac{1}{r} = \frac{1 + \epsilon \cos \alpha\theta}{a(1-\epsilon^2)} \ ,$$

which is an ellipse for $\alpha = 1$, but is a *precessing* ellipse for $\alpha \neq 1$.

(b)

The precessing motion may be described in terms of the *rate of precession of the perihelion*, where the term perihelion is used (loosely) to denote *any* of the turning points of the orbit. Derive an approximate expression for the rate of precession when α is close to unity, in terms of the dimensionless quantity $\eta = C/ka$.

(c)

The ratio η is a measure of the strength of the perturbing inverse cube term relative to the main inverse square term of the force. Show that the rate of precession of Mercury's perihelion ($40''$ of arc per century) could be accounted for *classically*, if $\eta = 1.42 \times 10^{-7}$. [Mercury's period and eccentricity are 0.24y and 0.206, respectively.]

7.

A He nucleus with velocity $v = 0.05c$ is normally incident on an Au foil that is 1 micron (1×10^{-6} m) thick. What is the probability that it will scatter into the backward hemisphere, *i.e.* bounce off the foil? (Please supply a *number*.)

8.

Calculate the differential cross section $d\sigma/d\Omega$ and the total cross section σ_T for the elastic scattering of a point particle from an impenetrable sphere; *i.e.*, the potential is given by $U(r) = 0,\ r > a$; $U(r) = \infty,\ r < a$.

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO PROBLEM SET 7
Solutions by J. Barber and T. Bunn

Reading:
105 Notes 8.1-8.3, 6.1-6.2 (again).
Hand & Finch 4.7, 5.1.

1.
[This a (hopefully clearer) version of Hand & Finch 4.17, "tetherball".] A mass m is attached to a weightless string that initially has a length s_0. The other end of the string is attached to a post of radius a. Neglect the effect of gravity. Suppose that the mass is set into motion. It is given an initial velocity of magnitude v_0 directed so that the string remains taut. The string wraps itself around the post, causing the mass to spiral inward toward it.

(**a**)
Write the Lagrangian in terms of $\dot{x}$ and $\dot{y}$, the cartesian velocity components of the mass. Is there a potential energy term?

Solution:

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right)$$

There is no potential energy term.

(**b**)
Use as generalized coordinates $s(t)$, the length of the part of the string that is not yet in contact with the post, and $\psi(t)$, the azimuthal angle at which the string barely fails to make contact with the post. Express $\dot{x}$ and $\dot{y}$ in terms of these generalized coordinates and their time derivatives.

Solution:
Letting the origin be at the center of the post, and letting ψ be the counter-clockwise angle from the x axis, we can write x and y as:

$$\begin{aligned} x &= a\cos\psi - s\sin\psi \\ y &= a\sin\psi + s\cos\psi \end{aligned}$$

Taking the time derivative of these expressions yields:

$$\begin{aligned} \dot{x} &= -a\dot{\psi}\sin\psi - \dot{s}\sin\psi - s\dot{\psi}\cos\psi \\ \dot{y} &= +a\dot{\psi}\cos\psi + \dot{s}\cos\psi - s\dot{\psi}\sin\psi \end{aligned}$$

(**c**)
Write a (constraint) equation relating $\dot{s}$ to $\dot{\psi}$. Use it to greatly simplify your answers for (**b**). Rewrite the Lagrangian using s as the only generalized coordinate.

Solution:
Since the string is winding up on the post, and thus decreasing its length, we must have:

$$\dot{s} = -a\dot{\psi}$$

Plugging this into our expressions for $\dot{x}$ and $\dot{y}$ gives us:

$$\begin{aligned} \dot{x} &= \frac{s\dot{s}}{a}\cos\psi \\ \dot{y} &= \frac{s\dot{s}}{a}\sin\psi \end{aligned}$$

And so the lagrangian becomes:

$$\mathcal{L} = \frac{m}{2a^2}s^2\dot{s}^2$$

(**d**)
Use the Euler-Lagrange equation to obtain an equation of motion for s. (You don't need to solve it.)

Solution:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{s}}\right) &= \frac{\partial\mathcal{L}}{\partial s} \\ \frac{d}{dt}\left(\frac{m}{a^2}s^2\dot{s}\right) &= \frac{m}{a^2}s\dot{s}^2 \\ s\ddot{s} + \dot{s}^2 &= 0 \end{aligned}$$

(e)
Since the Lagrangian has no explicit time dependence, and it depends quadratically on $\dot{s}$, the total energy is conserved. Write an equation setting the initial energy (expressed in terms of v_0) equal to the energy at an arbitrary value of s (expressed in terms of s and $\dot{s}$).
Solution:

$$\begin{aligned} E = \mathcal{H} &= \frac{\partial \mathcal{L}}{\partial \dot{s}}\dot{s} - \mathcal{L} \\ &= \frac{m}{2a^2}s^2\dot{s}^2 = \frac{m}{2}v_o^2 \end{aligned}$$

(f)
Use this equation to express dt in terms of ds multiplied by a function of s. Integrate it to solve for the time T that elapses before the mass hits the post. You should obtain the simple result

$$T = \frac{s_0^2}{2av_0} \ .$$

Solution:

$$\begin{aligned} \dot{s}^2 &= \frac{a^2v_o^2}{s^2} \\ \dot{s} &= -\frac{av_o}{s} \\ dt &= -\frac{s}{av_o}ds \end{aligned}$$

$$\begin{aligned} T &= -\int_{s_o}^{0} \frac{s}{av_o}ds \\ &= \frac{1}{av_o}\int_0^{s_o} s\ ds \\ &= \frac{s_o^2}{2av_o} \end{aligned}$$

Note that this could also be found by realizing that the DE in part (d) can also be written as $\frac{d}{dt}(s\dot{s}) = 0$. This, along with the initial conditions that $s(0) = s_o$ and $\dot{s}(0) = -\frac{av_o}{s_o}$ (obtainable from the energy expression), allows us to show that:

$$s(t) = \sqrt{s_o^2 - 2av_ot}$$

(g)
Is the angular momentum of the mass about the *axis* of the post conserved in this problem? Why or why not?
Solution:
Since the mass is constrained to lie in the xy plane, there is only a z component of angular momentum:

$$\begin{aligned} L_z &= m(x\dot{y} - y\dot{x}) \\ &= -\frac{ms^2\dot{s}}{a} \end{aligned}$$

whose magnitude is diminishing with time. Angular momentum is not conserved because the string is not directed toward the center of the post. Hence the tension in the string exerts a *torque* on the mass with respect to the post's center.

2.
Hand & Finch 4.19.
Solution:
(a)

$$\tau = 2\pi\sqrt{\frac{\mu}{k}}a^{\frac{3}{2}} \ \ \text{(Hand \& Finch Eq. 4.61)}\ ,$$

where $k = GM_sm_e$, $\mu \approx m_e$, and $a = R_e$. Also, $\tau = 1$ year $= 3.15 \times 10^7$ s. This allows us to solve for M_s:

$$\begin{aligned} M_s &= \frac{4\pi^2{R_e}^3}{G\tau^2} \\ &= 1.97 \times 10^{30}\ \text{kg} \end{aligned}$$

(b)

$$\begin{aligned} m_e &= \frac{4\pi^2{R_m}^3}{G{\tau_m}^2} \\ \frac{M_s}{m_e} &= \frac{{R_e}^3}{{\tau_e}^2}\frac{{\tau_m}^2}{{R_m}^3} \\ &= 3.38 \times 10^5 \\ m_e &= M_s\frac{m_e}{M_s} \\ &= 5.83 \times 10^{24}\ \text{kg} \end{aligned}$$

This is close to the actual value of 5.98×10^{24} kg. The density of the earth is

$$\begin{aligned} \rho_e &= \frac{m_e}{\frac{4\pi}{3}{r_e}^3} \\ &= 5.36 \times 10^3\ \frac{\text{kg}}{\text{m}^3} = 5.36\ \frac{\text{g}}{\text{cm}^3} \end{aligned}$$

The mass of the moon (1.23% of the earth's mass) could be similarly determined by observing the effects of its gravity on other objects, such as spacecraft, orbiting around it. Also, one could compare the height of the tides when the sun is aligned *vs.* anti-aligned with the moon: knowing the radii of the moon's and earth's orbits from their periods, one can solve for the ratio of the moon's and sun's masses (see problem 3). Finally, the moon's mass may be measured from its perturbations on the earth's orbit – but this is a dense topic.

3.
Hand & Finch 4.21.
Solution:
(**a**)
To quote Feynman: "The pull of the Moon for the Earth and for the water is 'balanced' at the center. But the water which is closer to the Moon is pulled *more* than the average and the water which is farther away from it is pulled *less* than the average. Furthermore, the water can flow while the more rigid Earth cannot..."(Feynman Lectures I). So the near water gets pulled away from the Earth, which in turn gets pulled away from the far water. This causes a net 'elongation' of the Earth and its oceans, directed approximately along the line joining the Earth and Moon. There is one high tide at the front of the Earth and one at the back.
(**b**)
Since the Earth is rotating, the direction of the elongation of the Earth is always changing with respect to the Earth's surface. The dual bulges of water on either side of the Earth cannot change their position instantly (because of viscous friction and the water's inertia) so there is a constant phase lag between the direction of elongation and the Earth-Moon direction. A similar tidal effect can be used to explain why only one side of the Moon ever faces the Earth.
(**c**)

$$
\begin{aligned}
F_{\text{tide, sun}} &= F_{s-e}(R_e - r_e) - F_{s-e}(R_e + r_e) \\
&\approx 2r_e \frac{\partial F_{s-e}}{\partial r}\Big|_{R_e} \\
&= \frac{4r_e G M_s m_e}{{R_e}^3} \\
F_{\text{tide, moon}} &\approx 2r_e \frac{\partial F_{m-e}}{\partial r}\Big|_{R_m} \\
&= \frac{4r_e G m_m m_e}{{R_m}^3} \\
\frac{F_{\text{tide, sun}}}{F_{\text{tide, moon}}} &= \frac{M_s}{{R_e}^3}\frac{{R_m}^3}{m_m} \\
&= 0.45
\end{aligned}
$$

The two tidal forces are of the same order of magnitude.

4.
Consider a particle of mass m that is constrained to move on the surface of a paraboloid whose equation (in cylindical coordinates) is $r^2 = 4az$. If the particle is subject to a gravitational force $-mg\hat{\mathbf{z}}$, show that the frequency of small oscillations about a circular orbit with radius $\rho = \sqrt{4az_0}$ is

$$\omega = \sqrt{\frac{2g}{a + z_0.}}$$

Solution:
In cylindrical coordinates (r, θ, z), we write the Lagrangian as

$$\mathcal{L} = \tfrac{1}{2} m\dot{r}^2 + \tfrac{1}{2} m r^2 \dot{\theta}^2 + \tfrac{1}{2} m\dot{z}^2 - mgz\ .$$

Use the equation of constraint $r^2 = 4az$ to get rid of z:

$$\mathcal{L} = \tfrac{1}{2} m\dot{r}^2 + \frac{m r^2 \dot{r}^2}{8a^2} + \tfrac{1}{2} m r^2 \dot{\theta}^2 - \frac{mgr^2}{4a}\ .$$

The Euler-Lagrange equation for θ just expresses angular momentum conservation: $l \equiv mr^2\dot{\theta}$ is constant. The equation for r is

$$\ddot{r}\left(1 + \frac{r^2}{4a^2}\right) + \frac{r\dot{r}^2}{4a^2} = \frac{l^2}{m^2 r^3} - \frac{gr}{2a}, \qquad (1)$$

where we have substituted $\dot{\theta} = l/mr^2$ and canceled a factor m. Choose ρ such that $r(t) = \rho$

is the constant-radius solution to this equation, and consider small perturbations about this solution: $r(t) = \rho + x(t)$, with $x \ll \rho$. Drop all terms with more than one power of x in our differential equation for r, and you find that

$$\ddot{x}\left(1+\frac{\rho^2}{4a^2}\right) = \frac{l^2}{m^2(\rho+x)^3} - \frac{g(\rho+x)}{2a}$$
$$= \frac{l^2}{m^2\rho^3}\left(1-\frac{3x}{\rho}\right) - \frac{g}{2a}(\rho+x)\ .$$

(The last step comes from a Taylor series expansion of $(\rho+x)^{-3}$.) Since $r(t) = \rho$ is a solution to equation (1), we know that $l^2/m^2\rho^3 = g\rho/2a$. (This comes from setting $\dot{r} = \ddot{r} = 0$ in equation (1).) Use this to simplify the differential equation for x:

$$\ddot{x}\left(1+\frac{\rho^2}{4a^2}\right) + \frac{2g}{a}x = 0$$

This is the equation for a harmonic oscillator with frequency

$$\omega = \sqrt{\frac{2g/a}{1+\rho^2/4a^2}} = \sqrt{\frac{2g}{a+z_0}}.$$

5.

An orbit that is almost circular can be considered to be a circular orbit to which a small perturbation has been applied. Take ρ to be the (unperturbed) circular orbit radius and define

$$g(r) = \frac{1}{\mu}\frac{\partial U(r)}{\partial r}\ ,$$

where μ is the reduced mass and U is an arbitrary potential. Set the radius $r = \rho + x$, where x is a small perturbation.

(**a**)

Starting from the differential equation for r and using the fact that the angular momentum l is constant, substitute $r = \rho + x$. Retaining terms only to first order in x, Taylor expand $g(r)$ about the point $r = \rho$, and show that x satisfies the differential equation

$$\ddot{x} + \left[\frac{3g(\rho)}{\rho} + g'(\rho)\right]x = 0\ ,$$

where $g'(\rho)$ is dg/dr evaluated at $r = \rho$.

Solution:

The Lagrangian for a particle moving in a central force field is

$$\mathcal{L} = \tfrac{1}{2}\mu\dot{r}^2 + \tfrac{1}{2}\mu r^2\dot{\theta}^2 - U(r)$$

where r and θ are polar coordinates. The Euler-Lagrange equation for θ says that $l = \mu r^2\dot{\theta}$ is constant, and the equation for r is

$$\mu\ddot{r} = \mu r\dot{\theta}^2 - U' = \frac{l^2}{\mu r^3} - \mu g(r)$$

Let ρ be the radius of the constant-r solution to this equation. Then ρ satisfies $l^2/\mu^2\rho^3 = g(\rho)$. Now we can look for solutions of the form $r(t) = \rho + x(t)$, with $x \ll \rho$. Dropping all terms of higher than first order in x, our differential equation becomes

$$\ddot{x} = \frac{l^2}{\mu^2\rho^3}\left(1-\frac{3x}{\rho}\right) - g(\rho) - xg'(\rho)$$

where we have done a Taylor expansion in x of $(\rho+x)^{-3}$ and $g(\rho+x)$. Now use our equation for ρ above to simplify this:

$$\ddot{x} = g(\rho)\left(1-\frac{3x}{\rho}\right) - g(\rho) - xg'(\rho)$$
$$= -\left(\frac{3g(\rho)}{\rho} + g'(\rho)\right)x$$

(**b**)

Taking the force law to be $F(r) = -kr^{-n}$, where n is an integer, show that the angle between two successive values of $r = r_{\max}$ (the "apsidal angle") is $2\pi/\sqrt{3-n}$. Thus, if $n > -6$, show that in general a closed orbit will result only for the harmonic oscillator force and the inverse square law force.

Solution:

$g(r) = \frac{k}{\mu}r^{-n}$, so $g'(r) = -\frac{k}{\mu}nr^{-(n+1)}$. Plug that into our equation for $\ddot{x}$ and you'll find that

$$\ddot{x} + \left(\frac{k}{\mu}\rho^{-(n+1)}(3-n)\right)x = 0.$$

This is the equation for a harmonic oscillator with frequency

$$\omega_x = \left(\frac{k}{\mu}\rho^{-(n+1)}(3-n)\right)^{1/2}$$

We need to compare this to ω_θ, the angular frequency of the unperturbed circular motion. Use the centripetal force equation $F_{\rm cent} = \mu\rho\dot{\theta}^2$ to get

$$\omega_\theta = \dot{\theta} = \left(\frac{U'}{\mu\rho}\right)^{1/2} = \left(\frac{k}{\mu}\rho^{-(n+1)}\right)^{1/2}$$

Successive maxima of r occur when $\omega_x t$ increases by 2π, and the angle through which the particle has moved in that time is

$$\Delta\theta = \omega_\theta t = 2\pi\frac{\omega_\theta}{\omega_x} = \frac{2\pi}{\sqrt{3-n}}$$

If $\Delta\theta/2\pi$ is a rational number, say j/k with j and k integers, then after j orbits the path will close. If $\Delta\theta$ is irrational, the orbit will never close. Looking at our expression for $\Delta\theta$, it's clear that the orbit closes only if $3-n$ is a perfect square, which only happens for $n > -6$ if $n = 2$ or $n = -1$.

6.
Consider the motion of a particle in a central force field $F = -k/r^2 + C/r^3$.
(**a**)
Show that the equation of the orbit can be put in the form

$$\frac{1}{r} = \frac{1+\epsilon\cos\alpha\theta}{a(1-\epsilon^2)}\ ,$$

which is an ellipse for $\alpha = 1$, but is a *precessing* ellipse for $\alpha \neq 1$.
Solution:
Start from Eq. (7.8) in the lecture notes:

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu k}{l^2}$$

Multiply it by u^2l^2/μ. Then, in place of the gravitational force $-ku^2$, substitute the full force $-ku^2 + Cu^3$. This yields

$$\frac{l^2u^2}{m}(u'' + u) = ku^2 - Cu^3\ ,$$

where u'' denotes $d^2u/d\theta^2$. Rearranging this equation, we get

$$u'' + \left(1 + \frac{mC}{l^2}\right)u - \frac{km}{l^2} = 0$$

This looks like the differential equation for a harmonic oscillator, plus a constant displacement. So we know it has a solution of the form $u = A + B\cos\alpha\theta$. Substituting this into the differential equation, we find that we get a valid solution as long as

$$\alpha = \sqrt{1+\frac{mC}{l^2}} \ \text{ and }\ A = \frac{km/l^2}{1+mC/l^2}$$

(B is arbitrary). Comparing this with the form given for a precessing ellipse:

$$u = \frac{1}{r} = \frac{1+\epsilon\cos\alpha\theta}{a(1-\epsilon^2)}\ ,$$

we find that $a(1-\epsilon^2) = 1/A$ and $a(1-\epsilon^2)/\epsilon = 1/B$. If you wanted to, you could solve these for a and ϵ, but there's no need to.
(**b**)
The precessing motion may be described in terms of the *rate of precession of the perihelion*, where the term perihelion is used (loosely) to denote *any* of the turning points of the orbit. Derive an approximate expression for the rate of precession when α is close to unity, in terms of the dimensionless quantity $\eta = C/ka$.
Solution:
How fast is the ellipse precessing? Well, between successive maxima of r, θ increases by $2\pi/\alpha$, and if the ellipse weren't precessing at all, that angle would be 2π. So the amount of precession per revolution is $\Delta\theta = 2\pi(1-1/\alpha)$. Now let's assume α is close to 1. That means that $mC/l^2 \ll 1$. In this approximation, we can expand $1/\alpha = (1+mC/l^2)^{-1/2} \approx 1 - mC/2l^2$. (This is just a Taylor expansion.) So the precession rate is

$$\Delta\theta = \frac{\pi mC}{l^2}$$

per orbit. To write this in terms of η, just note from above that

$$a(1-\epsilon^2) = \frac{l^2}{km}(1+mC/l^2) \approx l^2/km \ .$$

(Why were we able to drop the mC/l^2? Because it is small compared to 1.) Rearranging this, we get

$$\frac{m}{l^2} = \frac{1}{ka(1-\epsilon^2)} \ ,$$

and plugging that into our expression for $\Delta\theta$, we get

$$\Delta\theta = \frac{\pi C}{ka(1-\epsilon^2)} = \frac{\pi\eta}{(1-\epsilon^2)} \ .$$

(**c**)
The ratio η is a measure of the strength of the perturbing inverse cube term relative to the main inverse square term of the force. Show that the rate of precession of Mercury's perihelion ($40''$ of arc per century) could be accounted for *classically*, if $\eta = 1.42 \times 10^{-7}$. [Mercury's period and eccentricity are 0.24 y and 0.206, respectively.]
Solution:
If $\eta = 1.42 \times 10^{-7}$ and $\epsilon = 0.206$, then $\Delta\theta = 4.66 \times 10^{-7}$ radians $= 9.61'' \times 10^{-2}$. That's the precession per orbit, so to get the precession per century, we need to multiply by the number of orbits per century, 100/0.24. Then we find that $\Delta\theta = 40''$/century.

7.
A He nucleus with velocity $v = 0.05c$ is normally incident on an Au foil that is 1 micron (1×10^{-6} m) thick. What is the probability that it will scatter into the backward hemisphere, *i.e.* bounce off the foil? (Please supply a *number*.)
Solution:
First, what is the cross section that will result in the scattering of a He nucleus into the back hemisphere from a *single* Au nucleus?

$$\begin{aligned}
\sigma_{back} &= \int_{\theta=\frac{\pi}{2}}^{\theta=\pi} \frac{d\sigma}{d\Omega} d\Omega \\
&= \left(\frac{Zze^2}{2\mu v_o^2}\right)^2 \int_{\frac{\pi}{2}}^{\pi} \frac{2\pi \sin\theta \, d\theta}{\sin^4\frac{\theta}{2}} \\
&= 4\pi \left(\frac{Zze^2}{2\mu v_o^2}\right)^2 \int_{\frac{\pi}{2}}^{\pi} \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2} \, d\theta}{\sin^4\frac{\theta}{2}} \\
&= 8\pi \left(\frac{Zze^2}{2\mu v_o^2}\right)^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos u \, du}{\sin^3 u} \quad (u = \frac{\theta}{2}) \\
&= \pi \left(\frac{Zze^2}{\mu v_o^2}\right)^2
\end{aligned}$$

We also need the number of Au nuclei per unit area that the He nucleus sees in the foil. With a little thought, it can be seen that

$$\frac{\text{number}}{\text{unit Area}} = \frac{\rho_{\text{Au}} t}{m_{\text{Au}}} \ ,$$

where t is the foil thickness. The probability of backwards scattering from the foil is

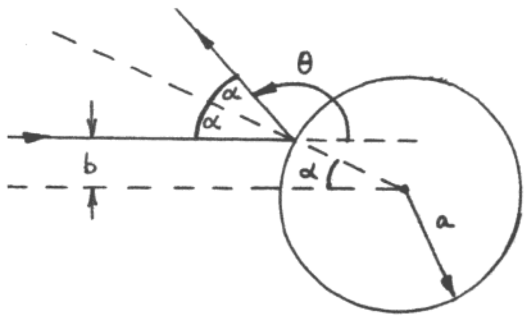
$$\begin{aligned}
P_{\text{back}} &= \frac{\sigma_{\text{back}}}{\text{Area}_{\text{foil}}} \times (\text{\# of Au nuclei in foil}) \\
&= \sigma_{\text{back}} \times \frac{\text{number}}{\text{unit Area}} \\
&= \frac{\pi \rho_{\text{Au}} t}{m_{\text{Au}}} \left(\frac{Zze^2}{\mu v_o^2}\right)^2
\end{aligned}$$

The numerical values (note we are working in cgs) of all these quantities are:

$$\begin{aligned}
\rho_{\text{Au}} &= 19.28 \ \frac{\text{g}}{\text{cm}^3} \\
t &= 10^{-4} \text{ cm} \\
m_{\text{Au}} &= 3.28 \times 10^{-22} \text{ g} \\
Z &= 79 \\
z &= 2 \\
e &= 4.8 \times 10^{-10} \text{ esu} \\
\mu &\approx m_{\text{He}} = 6.67 \times 10^{-24} \text{ g} \\
v_o &= .05 \ c = 1.5 \times 10^9 \ \frac{\text{cm}}{\text{s}}
\end{aligned}$$

Plugging in these values yields $P_{\text{back}} = 0.00011$, so most of the He nuclei do pass through the foil.

8.
Calculate the differential cross section $d\sigma/d\Omega$ and the total cross section σ_T for the elastic scattering of a point particle from an impenetrable sphere; *i.e.* the potential is given by $U(r) = 0,\ r > a$; $U(r) = \infty,\ r < a$.



Solution:
Consider a single particle approaching with impact parameter b and bouncing off of the sphere. Let θ be the scattering angle and let α be the angle the particle's trajectory makes with the normal to the sphere. The angle of reflection off the sphere equals the angle of incidence α. Therefore $2\alpha + \theta = \pi$. Also, the angle θ is related to b by

$$\begin{aligned} b &= a \sin\alpha \\ &= a \sin\frac{\pi - \theta}{2} \\ &= a\cos\frac{\theta}{2} \end{aligned}$$

The differential cross section is given by

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{b(\theta)}{\sin\theta}\left|\frac{db}{d\theta}\right| \\ &= \frac{a\cos\frac{\theta}{2}}{\sin\theta}\left|-\frac{a}{2}\sin\frac{\theta}{2}\right| \\ &= \frac{a^2}{4} \end{aligned}$$

So $\frac{d\sigma}{d\Omega}$ is constant. The total cross section is

$$\begin{aligned} \sigma &= \int \frac{d\sigma}{d\Omega}\, d\Omega \\ &= \frac{a^2}{4}\int_0^\pi 2\pi\ \sin\theta\, d\theta = \pi a^2 \end{aligned}$$

This is the cross-sectional area of the sphere, which is what we should expect.

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

ASSIGNMENT 8

Reading:
105 Notes 9.1-9.8
Hand & Finch 7.1-7.10, 8.1-8.3

1.
Discuss the implications of Liouville's theorem on the focussing of beams of charged particles by considering the following simple case. An electron beam of circular cross section (radius R_0) is directed along the z axis. The density of electrons along the beam is constant, and the electrons all have the same z momenta, but their much smaller momentum components transverse to the beam (p_x or p_y) are distributed uniformly over a circle of radius p_0 in momentum space. If some focussing system is used to reduce the beam radius from R_0 to R_1, find the resulting distribution of the transverse momentum components. What is the physical meaning of this result? (Consider the angular divergence of the beam.)

2.
In New Orleans (30° N latitude), there is a hockey arena with frictionless ice. The ice was formed by flooding a rink with water and allowing it to freeze slowly. This implies that a plumb bob would always hang in a direction perpendicular to the small patch of ice directly beneath it.

Show that a hockey puck (shot slowly enough that it stays in the rink!) will travel in a *circle*, making one revolution every day.

3.
Consider a situation exactly the same as in the previous problem, except that the rink is centered at the *north pole*. This stimulates a controversy:

Simplicio: "The angular frequency of circular motion of the puck is $2\Omega_e \cos\lambda$ with $\cos\lambda = 1$ rather than $\frac{1}{2}$ as in the previous problem [where Ω_e is the angular velocity of the earth's rotation about its axis]. So $\omega_{\text{puck}} = 2\Omega_e$."

Salviati: "Work the problem in the [inertial] reference frame of the fixed stars. For a particular set of initial conditions, the puck can be motionless in this frame while the earth and rink rotate under it. Then $\omega_{\text{puck}} = \Omega_e$."

Who is right? Why?

4.
Consider a particle that is projected vertically upward from a point on the earth's surface at north latitude ψ_0 (measured from the equator). (Here "upward" means opposite to the direction that a plumb bob hangs.) Show that it strikes the ground at a point $\frac{4}{3}\omega\sqrt{(8h^3/g)}\cos\psi_0$ to the west, where ω is the earth's angular velocity and h is the height reached. [*Hints*: Neglect air resistance and consider only heights small enough that g remains constant. Simplify your algebra by using the fact that the Coriolis force is very small with respect to the gravitational force – more quantitatively $\omega T \ll 1$, where T is the flight's duration.]

5.
Consider the description of the motion of a particle in a coordinate system that is rotating with uniform angular velocity ω with respect to an inertial reference frame. Use cylindrical coordinates, taking $\hat{z}$ to lie along the axis of rotation, and assume that the ordinary potential energy U is velocity-independent. Obtain the Lagrangian for the particle in the rotating system. Calculate the Hamiltonian and identify this quantity with the total energy E. Show that $E = \frac{1}{2}mv^2 + U + U'$, where U is the ordinary potential energy and U' is a pseudopotential. How does U' depend on the cylindrical coordinate r?

6.
Consider an Euler rotation

$$\begin{aligned}\tilde{x} &= \Lambda_3 \tilde{x}''' \\ &= \Lambda_3 \Lambda_2 \tilde{x}'' \\ &= \Lambda_3 \Lambda_2 \Lambda_1 \tilde{x}' \ ,\end{aligned}$$

where $\tilde{x}$ is a vector in the body axes and $\tilde{x}'$ is a vector in the space axes. Here

$$\Lambda_1 \equiv \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\Lambda_2 \equiv \begin{pmatrix} 0 & 0 & 1 \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{pmatrix}$$
$$\Lambda_3 \equiv \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

In the body axes, define

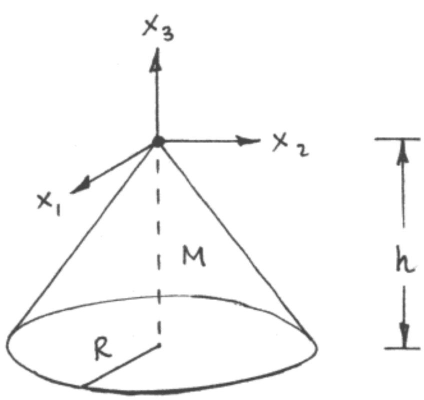
$$\vec{\omega} = \vec{\omega}_\phi + \vec{\omega}_\theta + \vec{\omega}_\psi \ ,$$

where

$$\begin{aligned}\vec{\omega}_\phi &\equiv \hat{x}_3' \dot{\phi} \\ \vec{\omega}_\theta &\equiv \hat{x}_1'' \dot{\theta} \\ \vec{\omega}_\psi &\equiv \hat{x}_3''' \dot{\psi} \ .\end{aligned}$$

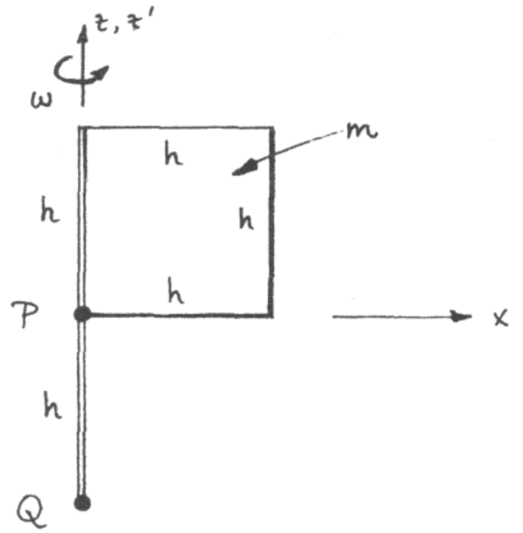
Find the components of $\vec{\omega}$ along the x_1', x_2', and x_3' (fixed) axes.

7.
Calculate the inertia tensor of a uniform right circular cone of mass M, radius R, and height h. Take the x_3 direction to be along the cone's axis. For this calculation, take the origin to be...



(**a**)
... at the apex of the cone (as shown in the figure).
(**b**)
... at the cone's center of mass.

8.
A square door of side h and mass m rotates with angular velocity ω about the z' (space) axis. The door is supported by a stiff light rod of length $2h$ which passes through bearings at points P and Q. P is at the origin of the primed (fixed) and unprimed (body) coordinates, which are coincident at $t = 0$. Neglect gravity.



(**a**)
Calculate the angular momentum **L** about P in the body system.
(**b**)
Transform to get $\mathbf{L}'(t)$ in the fixed system.
(**c**)
Find the torque $\mathbf{N}'(t)$ exerted about the point P by the bearings.
(**d**)
Assuming that the bearing at P exerts no torque about P, find the force $\mathbf{F}_Q'(t)$ exerted by the bearing at Q.

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO PROBLEM SET 8

Solutions by T. Bunn and M. Strovink

Reading:
105 Notes 9.1-9.8
Hand & Finch 7.1-7.10, 8.1-8.3

1.
Discuss the implications of Liouville's theorem on the focussing of beams of charged particles by considering the following simple case. An electron beam of circular cross section (radius R_0) is directed along the z axis. The density of electrons along the beam is constant, and the electrons all have the same z momenta, but their much smaller momentum components transverse to the beam (p_x or p_y) are distributed uniformly over a circle of radius p_0 in momentum space. If some focussing system is used to reduce the beam radius from R_0 to R_1, find the resulting distribution of the transverse momentum components. What is the physical meaning of this result? (Consider the angular divergence of the beam.)

Solution:
Since all the particles in the beam have the same z momenta, only four of the six dimensions of phase space are of interest: x, y, p_x, and p_y. The particles all lie in a circle in the x-y plane of radius R_0, and in a circle in the p_x-p_y plane of radius p_0. So the total volume in the four-dimensional phase space is the product of the areas of the circles: $V = \pi^2 R_0^2 p_0^2$. If we shrink the x-y circle from R_0 to R_1, we have to increase the radius of the momentum circle by the same factor to keep V constant. So the beam will now fill a circle in momentum space of radius $p_1 = p_0 \frac{R_0}{R_1}$. The ratio p_0/p_z is a measure of the angular divergence of the beam, so this result says that if you make the physical size of the beam smaller, you force it to become less well collimated, i.e. more divergent.

2.
In New Orleans (30° N latitude), there is a hockey arena with frictionless ice. The ice was formed by flooding a rink with water and allowing it to freeze slowly. This implies that a plumb bob would always hang in a direction perpendicular to the small patch of ice directly beneath it.

Show that a hockey puck (shot slowly enough that it stays in the rink!) will travel in a *circle*, making one revolution every day.

Solution:
The significance of the fact that the ice is perpendicular to a plumb bob is that there is no component of the combined (gravitational + centrifugal) force that is not cancelled by the normal force from the ice. (See the solution to Problem 4 for additional discussion of this point.)

Take ($\hat{z}$ = up [opposite to plumb bob direction], $\hat{y}$ = north, $\hat{x}$ = east) immediately above a patch of ice at latitude 30° (colatitude $\lambda \equiv 60°$). In this system, the angular velocity of the earth's rotation is directed along

$$\hat{\omega}_e = \hat{z}\cos\lambda + \hat{y}\sin\lambda \ .$$

If its instantaneous (horizontal) velocity is $\vec{v}$, with components v_x and v_y, a puck of mass m feels a Coriolis force

$$\vec{F}_{\rm Cor} = -2m\vec{\omega}_e \times \vec{v} \ .$$

The horizontal component of the force is

$$\begin{aligned} \vec{F}^{xy}_{\rm Cor} &= 2m(\hat{y}\,\omega_{ez}v_x - \hat{x}\,\omega_{ez}v_y) \\ &\equiv m\vec{\Omega} \times \vec{v} \quad \text{where} \\ \vec{\Omega} &\equiv -2\hat{z}\,\omega_{ez} \\ &= -2\hat{z}\,\Omega_e\cos\lambda \ . \end{aligned}$$

The effect of the horizontal component of the Coriolis force on the puck is

$$\begin{aligned} m\dot{\vec{v}} &= \vec{F}^{xy}_{\rm Cor} \\ &= m\vec{\Omega} \times \vec{v} \ . \end{aligned}$$

When the rate of change of a vector is always perpendicular to its instantaneous direction, the vector's length stays fixed. The last equation requires its direction to precess about $\vec{\Omega}$ with an angular velocity equal to the magnitude of Ω. Since $\vec{\Omega}$ is directed along $-\hat{z}$, the precession is clockwise. The period is

$$\frac{2\pi}{\Omega} = \frac{2\pi}{2\Omega_e \cos\lambda} = \frac{2\pi}{\Omega_e} = 1 \text{ day.}$$

3.
Consider a situation exactly the same as in the previous problem, except that the rink is centered at the *north pole*. This stimulates a controversy:

Simplicio: "The angular frequency of circular motion of the puck is $2\Omega_e \cos\lambda$ with $\cos\lambda = 1$ rather than $\frac{1}{2}$ as in the previous problem [where Ω_e is the angular velocity of the earth's rotation about its axis]. So $\omega_{\text{puck}} = 2\Omega_e$."

Salviati: "Work the problem in the [inertial] reference frame of the fixed stars. For a particular set of initial conditions, the puck can be motionless in this frame while the earth and rink rotate under it. Then $\omega_{\text{puck}} = \Omega_e$."

Who is right? Why?
Solution:
Simplicio is right. Even if the rink is at the north pole, the general method of analysis used in the previous problem is still valid.

What confuses Salviati? What he has in mind is a hypothetical case in which the puck remains at rest with respect to the fixed stars (it is OK to neglect the fact that the earth orbits the sun). The puck lies at a fixed high latitude, a short distance from the north pole, so that it remains on the ice of a rink that is centered at the pole. As the earth turns under the puck, the rink rotates under it as well. This occurs with a period of one day, not half a day as Simplicio calculates.

To prove Salviati wrong, we need to show that his hypothetical case can't occur. Once again, the key is the definition of "up", which is directly related to the shape of the ice. Salviati presumes that the puck can remain at rest with respect to the fixed stars. Therefore (in the frame of the fixed stars) there can be no net force on it. In order for this to happen, the earth's gravitational force must exactly balance the normal force from the ice. Therefore the surface of the ice must have a spherical shape, with a radius of curvature equal to the distance to the earth's center.

However, we are told that the ice is formed by flooding the rink and allowing the water to freeze. This requires the surface of a patch of ice to be normal to a plumb bob held over it. Because the plumb bob feels both the earth's gravitational force and a centrifugal force corresponding to its rotation, the plumb bob will point to a location south of the earth's center. This will cause the radius of curvature of the ice at the north pole rink to be larger than the earth's radius. Because the ice has this shallower curvature, a small component of the normal force on the puck will not be offset by gravity and will push the puck toward the pole. In order to stay at the same latitude, it would be necessary for the puck to travel in a circle (as viewed in the fixed stars!) so that it would have enough poleward acceleration to obey this poleward force. The combination of this absolute circular motion, and the rotation of the earth under it, produces a net rotation of the puck relative to the earth which is twice as fast as Salviati envisages.

4.
Consider a particle that is projected vertically upward from a point on the earth's surface at north latitude ψ_0 (measured from the equator). (Here "upward" means opposite to the direction that a plumb bob hangs.) Show that it strikes the ground at a point $\frac{4}{3}\omega\sqrt{(8h^3/g)}\cos\psi_0$ to the west, where ω is the earth's angular velocity and h is the height reached. [*Hints*: Neglect air resistance and consider only heights small enough that g remains constant. Simplify your algebra by using the fact that the Coriolis force is very small with respect to the gravitational force – more quantitatively $\omega T \ll 1$, where T is the flight's duration.]
Solution:
First, let's make sure that we understand why "vertically upward" is taken to be opposite to the

direction of a plumb bob. Let $\mathbf{g}_{\text{grav}}$ be the gravitational acceleration vector. A mass hanging on a string will not align itself parallel to $\mathbf{g}_{\text{grav}}$, because it will feel the centrifugal force as well as the gravitational force. The total (gravitational plus centrifugal) acceleration on a stationary object is $\vec{g} = \mathbf{g}_{\text{grav}} - \vec{\omega} \times (\vec{\omega} \times \vec{r})$. So, if it were not already specified in the problem, we could imagine that the "vertical" direction could be chosen to be parallel either to $\mathbf{g}_{\text{grav}}$ or to $\vec{g}$. If the latter is specified, than we can ignore centrifugal forces for the rest of the problem, since the gravitational plus centrifugal force has no component perpendicular to the "vertical" direction. (Note that the centrifugal force is of order 10^{-3} times the gravitational force, so the two directions differ by only a fraction of a degree, and $\vec{g}$ and $\mathbf{g}_{\text{grav}}$ are nearly equal in magnitude.)

So we choose a coordinate system as follows: Let the z-direction be vertically up in the sense just discussed (including the centrifugal force along with the gravitational force). Then let the y-direction be west (tangent to the earth's surface), and the x-direction be perpendicular to both z and y. (So x points mostly north.) Then $\vec{g} = -g\hat{z}$, and the earth's angular velocity is $\vec{\omega} = \omega\,(\hat{x}\cos\psi_0 + \hat{z}\sin\psi_0)$. Our particle is subject to the acceleration $\vec{g}$ and the Coriolis acceleration:

$$\ddot{\vec{r}} = \vec{g} - 2\vec{\omega} \times \vec{v}$$

In cartesian components, these equations are

$$\begin{aligned} \ddot{x} &= 2\omega_z \dot{y} \\ \ddot{y} &= 2\omega_x \dot{z} - 2\omega_z \dot{x} \\ \ddot{z} &= -g - 2\omega_x \dot{y} \end{aligned}$$

and we have initial conditions $x = y = z = \dot{x} = \dot{y} = 0$, $\dot{z} = v_0$. The three differential equations can all be integrated once to yield

$$\begin{aligned} \dot{x} &= 2\omega_z y \\ \dot{y} &= 2\omega_x z - 2\omega_z x \\ \dot{z} &= v_0 - gt - 2\omega_x y \end{aligned}$$

It's not so easy to solve these equations exactly, so let's use the method of perturbations. (Perturbative methods look promising, since we know that the actual solution to the equations is going to be extremely close to the solution we get by neglecting the Coriolis force.) The Coriolis force is weak, which means that we can treat ω as a small quantity, and expand in powers of it. Let $x_0(t), y_0(t), z_0(t)$ be the solutions to these equations to zeroth order in ω. These quantities are just the solutions to the equations of a particle moving under the influence of ordinary gravity: With our initial conditions, the solutions are

$$x_0(t) = 0 \qquad y_0(t) = 0 \qquad z_0(t) = v_0 t - \tfrac{1}{2} g t^2$$

Our solution is a small perturbation of this, so we can set $x(t) = x_0(t) + x_1(t)$, along with similar expressions for y and z. Then x_1, y_1, and z_1 are all small. We shall retain only terms to first order in the small quantities ω, x_1, y_1, z_1. First we write the equation for $\dot{x}$:

$$\dot{x}_1 = 2\omega_z y_1 \ ,$$

where we have used the fact that $x_0 = y_0 = 0$. The right-hand side is second order in the small quantities. Neglecting it, we conclude that $x_1 \approx 0$. Therefore we're interested mainly in the westward displacement y_1. Let's write the equation for $\dot{y}$:

$$\dot{y}_1 = 2\omega_x z_0 - 2\omega_z x_0 = 2\omega_x (v_0 t - \tfrac{1}{2} g t^2)$$

Integrate this equation to get

$$y(t) = \omega_x \left(v_0 t^2 - \tfrac{1}{3} g t^3 \right)$$

From the z_0 equation, we find that the particle hits the ground at a time $T = 2v_0/g$, and the height is $h = v_0^2/2g$. So the westerly displacement is

$$y_1(T) = \frac{4}{3}\omega_x \left(\frac{8h^3}{g} \right)^{1/2} = \frac{4}{3}\omega \cos\psi_0 \left(\frac{8h^3}{g} \right)^{1/2}$$

5.

Consider the description of the motion of a particle in a coordinate system that is rotating with uniform angular velocity ω with respect to an inertial reference frame. Use cylindrical coordinates, taking $\hat{z}$ to lie along the axis of rotation, and assume that the ordinary potential energy U is velocity-independent. Obtain the Lagrangian for the particle in the rotating system. Calculate the Hamiltonian and identify this quantity with the total energy E. Show that $E = \frac{1}{2}mv^2 + U + U'$, where U is the ordinary potential energy and U' is a pseudopotential. How does U' depend on the cylindrical coordinate r?

Solution:

The velocity of a particle in a rotating coordinate system is $\vec{v} = \dot{\vec{r}} + \vec{\omega} \times \vec{r}$. Let's use cylindrical coordinates r, ϕ, z, with ω in the z-direction. Then $\vec{\omega} \times \vec{r} = \omega r \hat{\phi}$, so $\vec{v} = \dot{r}\hat{r} + \dot{z}\hat{z} + r(\dot{\phi} + \omega)\hat{\phi}$. The Lagrangian is therefore

$$\mathcal{L} = \tfrac{1}{2}mv^2 - U = \tfrac{1}{2}m\left(\dot{r}^2 + r^2(\dot{\phi}+\omega)^2 + \dot{z}^2\right) - U.$$

The canonical momenta are defined by $p_j = \partial\mathcal{L}/\partial\dot{q}_j$, so

$$p_r = m\dot{r} \qquad p_\phi = mr^2(\dot{\phi}+\omega) \qquad p_z = m\dot{z}$$

Use the definition of the Hamiltonian

$$\begin{aligned}
\mathcal{H} &= p_j\dot{q}_j - \mathcal{L} \\
&= m\left(\dot{r}^2 + r^2(\dot{\phi}+\omega)\dot{\phi} + \dot{z}^2\right) - \mathcal{L} \\
&= \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) - \tfrac{1}{2}mr^2\omega^2 + U \ .
\end{aligned}$$

The first term is just the usual expression for kinetic energy, so it represents the "apparent kinetic energy" observed by someone in the rotating coordinate system. The second term is the effective potential, and the last term is the ordinary potential.

6.

Consider an Euler rotation

$$\begin{aligned}
\tilde{x} &= \Lambda_\psi^t \tilde{x}''' \\
&= \Lambda_\psi^t \Lambda_\theta^t \tilde{x}'' \\
&= \Lambda_\psi^t \Lambda_\theta^t \Lambda_\phi^t \tilde{x}' \ ,
\end{aligned}$$

where $\tilde{x}$ is a vector in the body axes and $\tilde{x}'$ is a vector in the space axes. Here

$$\Lambda_\phi^t \equiv \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda_\theta^t \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$\Lambda_\psi^t \equiv \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

In the body axes, define

$$\vec{\omega} = \vec{\omega}_\phi + \vec{\omega}_\theta + \vec{\omega}_\psi \ ,$$

where

$$\begin{aligned}
\vec{\omega}_\phi &\equiv \hat{x}_3'\dot{\phi} \\
\vec{\omega}_\theta &\equiv \hat{x}_1''\dot{\theta} \\
\vec{\omega}_\psi &\equiv \hat{x}_3'''\dot{\psi} \ .
\end{aligned}$$

Find the components of $\vec{\omega}$ along the x_1', x_2', and x_3' (fixed) axes.

Solution:

At the outset, let's introduce some notation. Define the vectors $\hat{p} = \hat{x}_3'$, $\hat{q} = \hat{x}_1''$, and $\hat{r} = \hat{x}_3'''$. Then

$$\vec{\omega} = \dot{\phi}\hat{p} + \dot{\theta}\hat{q} + \dot{\psi}\hat{r}.$$

As a warmup, we write the components of each of the vectors $\hat{p}, \hat{q}, \hat{r}$ in the *unprimed* (body) coordinate system. Expressed in the primed (space) system, $\hat{p}'$ is $(0,0,1)$, so in the body system

$$\hat{p} = \Lambda_\psi^t \Lambda_\theta^t \Lambda_\phi^t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta\sin\psi \\ \sin\theta\cos\psi \\ \cos\theta \end{pmatrix}$$

(We have used the explicit forms of the Λ-matrices.) Similarly,

$$\hat{q} = \Lambda_\psi^t \Lambda_\theta^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\psi \\ -\sin\psi \\ 0 \end{pmatrix}$$

$$\hat{r} = \Lambda_\psi^t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .$$

Substituting these into our expression for $\vec{\omega}$, we get

$$\vec{\omega} = \begin{pmatrix} \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix}$$

in the *unprimed* basis. So far this parallels Lecture Notes section 9.3.

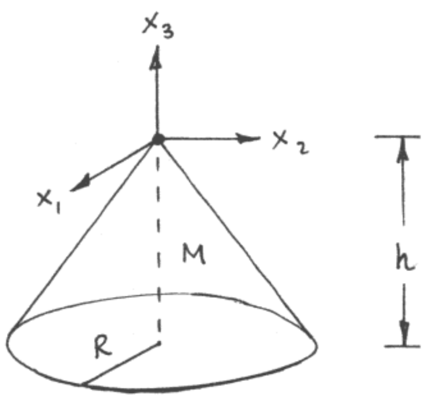
What we really need to do is to write $\hat{p}, \hat{q}, \hat{r}$ in the *primed* (fixed) basis. The first one is easy: $\hat{p}' = (0,0,1)$. Using the fact that the transformation matrices are orthogonal ($\Lambda_i^{-1} = \Lambda_i^t$), we get

$$\hat{q}' = \Lambda_\phi \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}$$
$$\hat{r}' = \Lambda_\phi \Lambda_\theta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\phi\sin\theta \\ -\sin\theta\cos\phi \\ \cos\theta \end{pmatrix} .$$

So, in the *primed* basis, the components of $\vec{\omega}'$ are

$$\vec{\omega}' = \begin{pmatrix} \dot\theta\cos\phi + \dot\psi\sin\phi\sin\theta \\ \dot\theta\sin\phi - \dot\psi\sin\theta\cos\phi \\ \dot\phi + \dot\psi\cos\theta \end{pmatrix}$$

7.
Calculate the inertia tensor of a uniform right circular cone of mass M, radius R, and height h. Take the x_3 direction to be along the cone's axis. For this calculation, take the origin to be...



(**a**)
... at the apex of the cone (as shown in the figure).
Solution:
Remember the formula

$$\mathcal{I}_{ij} = \int \rho(\vec{x}) \left(\delta_{ij}\vec{x}^2 - x_i x_j \right) d^3x$$

where $\rho(\vec{x})$ is the density at point $\vec{x}$. (In our case, ρ is just a constant.) All of the off-diagonal elements in our case are zero. To see why, consider $\mathcal{I}_{13}$:

$$\mathcal{I}_{13} = -\rho \int xz\, dx\, dy\, dz$$

But the region of integration is symmetric under the substitution $x \to -x$ (reflection about the yz plane), while the integrand changes sign under this reflection. So $\mathcal{I}_{13} = -\mathcal{I}_{13}$, which can only happen if $\mathcal{I}_{13} = 0$. All the off-diagonal elements of the inertia tensor vanish in the same way. (If you don't like this argument, you can just do the integrals explicitly.)

Now we need to compute the three diagonal elements. Let's start with $\mathcal{I}_{33}$. In cylindrical coordinates r, ϕ, z, for convenience directing z along $-x_3$, the integral is

$$\begin{aligned}
\mathcal{I}_{33} &= \rho \int \left(\vec{x}^2 - z^2 \right) r\, dr\, d\phi\, dz \\
&= \rho \int_0^h dz \int_0^{Rz/h} dr \int_0^{2\pi} d\phi\, r^3 \\
&= 2\pi\rho \int_0^h dz \tfrac{1}{4} \left(\frac{Rz}{h} \right)^4 \\
&= \frac{\pi}{10} \rho R^4 h
\end{aligned}$$

The volume of a cone is $\frac{\pi}{3}R^2 h$, so $\rho = 3M/\pi R^2 h$. Thus

$$\mathcal{I}_{33} = \tfrac{3}{10} M R^2$$

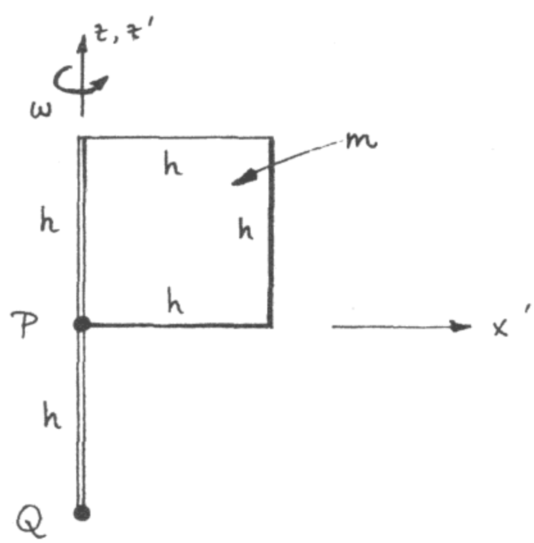
Now let's find $\mathcal{I}_{11}$:

$$\begin{aligned}
\mathcal{I}_{11} &= \rho \int \left(y^2 + z^2 \right) dx\, dy\, dz \\
&= \rho \int \left(r^2 \sin^2\phi + z^2 \right) r\, dr\, d\phi\, dz \\
&= \rho \int_0^h dz \int_0^{Rz/h} dr \int_0^{2\pi} d\phi\, r \left(r^2 \sin^2\phi + z^2 \right) \\
&= \pi\rho \int_0^h dz \int_0^{Rz/h} dr (r^3 + 2rz^2) \\
&= \frac{\pi\rho}{20} \left(R^4 h + 4R^2 h^3 \right) \\
&= \tfrac{3}{20} M \left(R^2 + 4h^2 \right) .
\end{aligned}$$

Of course, $\mathcal{I}_{11} = \mathcal{I}_{22}$ by symmetry.
(**b**)
... at the cone's center of mass.
Solution:
The parallel axis theorem says that moments of inertia about an axis that is displaced by a distance d from one of the principal axes passing through the center of mass is greater than the corresponding moment about the center of mass by the amount Md^2. In our case, the center of mass is a distance $d = \frac{3}{4}h$ in the z-direction away from the apex. So $\mathcal{I}_{33}$ is unaffected by moving to the center of mass, while $\mathcal{I}_{11}$ and $\mathcal{I}_{22}$ decrease by Md^2. That is,

$$\begin{aligned}\mathcal{I}_{11} = \mathcal{I}_{22} &= \tfrac{3}{20}M\left(R^2+4h^2\right) - \tfrac{9}{16}Mh^2 \\ &= \tfrac{3}{20}M\left(R^2+\tfrac{1}{4}h^2\right) \\ \mathcal{I}_{33} &= \tfrac{3}{10}MR^2\ .\end{aligned}$$

8.
A square door of side h and mass m rotates with angular velocity ω about the z' (space) axis. The door is supported by a stiff light rod of length $2h$ which passes through bearings at points P and Q. P is at the origin of the primed (fixed) and unprimed (body) coordinates, which are coincident at $t = 0$. Neglect gravity.



(**a**)
Calculate the angular momentum $\mathbf{L}$ about P in the body system.
Solution:
In the body axes, let's find the elements of the inertia tensor about the point P. The surface density is $\rho = m/h^2$, and all of our integrals need only be taken over x and z, with $y = 0$.

$$\begin{aligned}\mathcal{I}_{11} &= \rho\int_0^h\int_0^h z^2\,dx\,dz = \tfrac{1}{3}mh^2 \\ \mathcal{I}_{33} &= \rho\int_0^h\int_0^h x^2\,dx\,dz = \tfrac{1}{3}mh^2 \\ \mathcal{I}_{22} &= \rho\int_0^h\int_0^h (x^2+z^2)\,dx\,dz = \tfrac{2}{3}mh^2 \\ \mathcal{I}_{12} &= -\rho\int_0^h\int_0^h xy\,dx\,dz = 0 \\ \mathcal{I}_{23} &= -\rho\int_0^h\int_0^h yz\,dx\,dz = 0 \\ \mathcal{I}_{13} &= -\rho\int_0^h\int_0^h xz\,dx\,dz = -\tfrac{1}{4}mh^2\ .\end{aligned}$$

Using the fact that $\vec{\omega} = \omega\hat{z}$,

$$\begin{aligned}\vec{L} &= \mathcal{I}\vec{\omega} \\ &= mh^2\omega\begin{pmatrix}\frac{1}{3} & 0 & -\frac{1}{4} \\ 0 & \frac{2}{3} & 0 \\ -\frac{1}{4} & 0 & \frac{1}{3}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} \\ &= mh^2\omega\begin{pmatrix}-\frac{1}{4}\\0\\\frac{1}{3}\end{pmatrix}\ .\end{aligned}$$

So $\vec{L} = mh^2\omega(-\frac{1}{4}\hat{x} + \frac{1}{3}\hat{z})$.
(**b**)
Transform to get $\mathbf{L}'(t)$ in the fixed system.
Solution:
To convert to fixed (space) axes, note that $\hat{z} = \hat{z}'$, and $\hat{x} = \hat{x}'\cos\omega t + \hat{y}'\sin\omega t$. So

$$\vec{L}' = mh^2\omega\big(-\tfrac{1}{4}(\hat{x}'\cos\omega t + \hat{y}'\sin\omega t) + \tfrac{1}{3}\hat{z}'\big)\ .$$

(**c**)
Find the torque $\mathbf{N}'(t)$ exerted about the point P by the bearings.
Solution:

$$\vec{N}' = \dot{\vec{L}}' = \tfrac{1}{4}mh^2\omega^2(\hat{x}'\sin\omega t - \hat{y}'\cos\omega t)\ .$$

(**d**)
Assuming that the bearing at P exerts no torque about P, find the force $\mathbf{F}'_Q(t)$ exerted by the

bearing at Q.
Solution:
Taking $\vec{r}'$ to be a vector from P to Q, the torque $\vec{N}' = \vec{r}' \times \vec{F}'$. So the force $\vec{F}'$ at point Q causes a torque

$$\vec{N}' = h(F'_{y'}\hat{x}' - F'_{x'}\hat{y}') \ .$$

Setting this equal to the torque from the previous part, we get

$$\vec{F}' = \tfrac{1}{4}mh\omega^2(\hat{x}'\cos\omega t + \hat{y}'\sin\omega t) \ .$$

(In the body axes, this force points in a constant $(\hat{x})$ direction.)

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

ASSIGNMENT 9

Reading:
105 Notes 10.1-10.3, 11.1-11.4
Hand & Finch 8.4-8.12

1.
Three equal point masses are located at $(a, 0, 0)$, $(0, 2a, 2a)$, and $(0, 2a, a)$. About the origin, find the principal moments of inertia and a set of principal axes.

2.
Consider a rigid body that is plane, *i.e.* it lies in the plane $z = 0$.
(**a**)
Prove that the z axis is a principal axis.
(**b**)
Prove that the diagonalized inertia tensor for this plane rigid body has the largest element equal to the sum of the two smaller elements.

3.
Design a solid right circular cylinder so that if it is rotated about *any* axis that passes through its center of mass, it will continue to rotate about that axis without wobbling.

4.
Assume that the earth is a rigid solid sphere that is rotating about an axis through the North Pole. At $t = 0$ a mountain of mass 10^{-9} the earth's mass is added at north latitude 45°. The mountain is added "at speed" so that the earth's angular velocity ω is the same before and immediately after the mountain's addition.

Describe the subsequent motion of the rotation axis with respect to the North Pole. What is the velocity of its intersection with the earth's surface, in miles per year?

5.
Assume that the earth is a rigid solid ellipsoid of revolution, rotating about its symmetry axis $\hat{\mathbf{x}}_3$, and that it has $1 - (I_2/I_3) = -0.0033$ (actually the earth *bulges* at the equator, so that this quantity is really positive). Two equal mountains are placed opposite each other on the equator "at speed", so that ω is the same immediately afterward. What fraction of the earth's mass must each mountain have in order to render the earth's rotation barely unstable with respect to small deviations of $\hat{\omega}$ from the $\hat{\mathbf{x}}_3$ axis?

6.
Consider an asymmetric body (principal moments $I_3 > I_2 > I_1$) initially rotating with $\vec{\omega}$ very close to the $\hat{x}_3$ axis.
(**a**)
Show that the projection of $\vec{\omega}(t)$ on the $\hat{x}_1 - \hat{x}_2$ plane describes an *ellipse*.
(**b**)
Calculate the ratio of the major and minor axes of the ellipse.

7.
Consider a heavy symmetrical top with one point fixed. Show that the *magnitude* of the top's angular momentum about the fixed point can be expressed as a function only of the constants of motion and the polar angle θ of the top's axis.

8.
Investigate the motion of the heavy symmetrical top with one point fixed for the case in which the axis of rotation is vertical (along $\hat{\mathbf{x}}_3$). Show that the motion is either stable or unstable depending on whether the quantity $4I_2Mhg/I_3^2\omega_3^2$ is less than or greater than unity. (If the top is set spinning in the stable configuration ("sleeping"), it will become unstable as friction gradually reduces the value of ω_3. This is a familiar childhood observation.)

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SOLUTION TO PROBLEM SET 9

Solutions by T. Bunn and J. Barber

Reading:
105 Notes 10.1-10.3, 11.1-11.4
Hand & Finch 8.4-8.12

1.
Three equal point masses are located at $(a, 0, 0)$, $(0, 2a, 2a)$, and $(0, 2a, a)$. About the origin, find the principal moments of inertia and a set of principal axes.
Solution:
For a number of point masses, the expression for the inertia tensor becomes

$$I_{ij} = \sum m_k \left(|\mathbf{x_k}|^2 \delta_{ij} - x_{ki} x_{kj} \right)$$

So:

$$\begin{aligned}
I_{xx} &= m\left(0^2 + 0^2 + (2a)^2 + (2a)^2 + (2a)^2 + a^2\right) \\
&= 13ma^2 \\
I_{yy} &= m\left(a^2 + 0^2 + 0^2 + (2a)^2 + 0^2 + a^2\right) \\
&= 6ma^2 \\
I_{zz} &= m\left(a^2 + 0^2 + 0^2 + (2a)^2 + 0^2 + (2a)^2\right) \\
&= 9ma^2 \\
I_{yx} = I_{xy} &= -m\left((a)(0) + (0)(2a) + (0)(2a)\right) \\
&= 0 \\
I_{zx} = I_{xz} &= -m\left((a)(0) + (0)(2a) + (0)(a)\right) \\
&= 0 \\
I_{zy} = I_{yz} &= -m\left((0)(0) + (2a)(2a) + (2a)(a)\right) \\
&= -6ma^2
\end{aligned}$$

So the full inertia tensor is

$$\mathcal{I} = ma^2 \begin{pmatrix} 13 & 0 & 0 \\ 0 & 6 & -6 \\ 0 & -6 & 9 \end{pmatrix}$$

The principal moments and the principal axes are the eigenvalues and eigenvectors of this matrix. Skipping the explicit calculation, the principal moments are $13ma^2$ and $\frac{3}{2}(5 \pm \sqrt{17})ma^2$, and the (unnormalized) eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \frac{1}{4}(1 \pm \sqrt{17}) \\ 1 \end{pmatrix}.$$

2.
Consider a rigid body that is plane, *i.e.* it lies in the plane $z = 0$.
(a)
Prove that the z axis is a principal axis.
Solution:
If the z axis is a principal axis, then $\hat{z}$ will be an eigenvector of the inertia tensor. Thus we need to show that $I_{xz} = I_{yz} = 0$. That way, $\mathcal{I}\hat{z} = I_{xz}\hat{x} + I_{yz}\hat{y} + I_{zz}\hat{z} = I_{zz}\hat{z}$. Suppose $\sigma(x, y)$ is the surface mass density, so that the volume mass density is $\rho(x, y, z) = \sigma(x, y)\delta(z)$. Then:

$$\begin{aligned}
I_{xz} &= -\int \rho(x, y, z)\, y\, z\, dx\, dy\, dz \\
&= -\int \sigma(x, y)\, dx\, dy \int z\, \delta(z)\, dz \\
&= 0
\end{aligned}$$

$I_{yz} = 0$ by a similar argument. Thus $\hat{z}$ is a principal axis.

(b)
Prove that the diagonalized inertia tensor for this plane rigid body has the largest element equal to the sum of the two smaller elements.
Solution:
Let's assume we've chosen a coordinate system

whose axes are the principal axes. Then the three moments of inertia are

$$
\begin{aligned}
I_{xx} &= \int \sigma(x,y)\, y^2\, dx\, dy \\
I_{yy} &= \int \sigma(x,y)\, x^2\, dx\, dy \\
I_{zz} &= \int \sigma(x,y)\, (x^2+y^2)\, dx\, dy \ = I_{xx} + I_{yy}
\end{aligned}
$$

3.

Design a solid right circular cylinder so that if it is rotated about *any* axis that passes through its center of mass, it will continue to rotate about that axis without wobbling.

Solution

We know from the Euler equations that if a body has all of its principal moments of inertia equal to one another, then the components of the angular velocity vector will be constant, ie the body will not wobble. Therefore, we must choose the height to width ratio to make these moments equal. Choose the origin to be at the center of mass of the cylinder, with the z axis along the cylinder's axis. Then, by symmetry, $I_{xx} = I_{yy}$, and all off diagonal elements vanish. Note that if the cylinder has mass m, height h, and radius R, then the density $\rho = \frac{m}{\pi R^2 h}$.

$$
\begin{aligned}
I_{zz} &= \int_{-h/2}^{h/2} dz \int_0^{2\pi} d\phi \int_0^R r\, dr\, \rho(x^2+y^2) \\
&= \frac{m}{\pi R^2 h} \int_{-h/2}^{h/2} dz \int_0^{2\pi} d\phi \int_0^R r^3\, dr \\
&= \frac{1}{2} m R^2 \\
I_{xx} &= \int_{-h/2}^{h/2} dz \int_0^{2\pi} d\phi \int_0^R r\, dr\, \rho(y^2+z^2) \\
&= \frac{m}{\pi R^2 h} \int_{-\frac{h}{2}}^{\frac{h}{2}} dz \int_0^{2\pi} d\phi \int_0^R r\, dr\, (r^2 \sin^2\phi + z^2) \\
&= \frac{1}{2} m \left(\frac{1}{2} R^2 + \frac{1}{6} h^2 \right)
\end{aligned}
$$

We must have

$$
\begin{aligned}
I_{yy} = I_{xx} &= I_{zz} \\
R^2 &= \frac{1}{2} R^2 + \frac{1}{6} h^2 \\
R &= \frac{h}{\sqrt{3}}
\end{aligned}
$$

4.

Assume that the earth is a rigid solid sphere that is rotating about an axis through the North Pole. At $t = 0$ a mountain of mass 10^{-9} the earth's mass is added at north latitude 45°. The mountain is added "at speed" so that the earth's angular velocity ω is the same before and immediately after the mountain's addition.

Describe the subsequent motion of the rotation axis with respect to the North Pole. What is the velocity of its intersection with the earth's surface, in miles per year?

Solution:

Let's choose our axes to be the principal axes of the whole system (earth plus mountain.) Specifically, let $\hat{x}_3$ point in the direction of the mountain, and let $\hat{x}_1$ point east and $\hat{x}_2$ point north. Then the moments of inertia about these axes are

$$
\begin{aligned}
I_1 = I_2 &= \frac{2}{5} M R^2 + m R^2 \\
I_3 &= \frac{2}{5} M R^2
\end{aligned}
$$

where M and R are the earth's mass and radius, and $m = 10^{-9} M$ is the mountain's mass. Let $\Delta I = I_1 - I_3$. Euler's equations are:

$$
\begin{aligned}
I_1 \dot{\omega}_1 &= \Delta I \omega_2 \omega_3 \\
I_1 \dot{\omega}_2 &= -\Delta I \omega_1 \omega_3 \\
I_3 \dot{\omega}_3 &= 0
\end{aligned}
$$

ω_3 is constant, by the third equation. If we define $\Omega = \Delta I \omega_3 / I_1$, then the first two equations become

$$
\begin{aligned}
\dot{\omega}_1 &= \Omega \omega_2 \\
\dot{\omega}_2 &= -\Omega \omega_1 \ .
\end{aligned}
$$

Combine them to get $\ddot{\omega}_1 = -\Omega^2 \omega_1$. This is the equation for a harmonic oscillator, so $\omega_1(t)$ is a linear combination of $\sin \Omega t$ and $\cos \Omega t$. Since $\omega_1(0) = 0$, we must have $\omega_1(t) = A \sin \Omega t$

for some A. Then, taking the time derivative, we get $\omega_2(t) = A\cos\Omega t$. At t=0, we have $\omega_2 = \omega_3 = \omega/\sqrt{2}$, so $A = \omega/\sqrt{2}$.

$$\vec{\omega}(t) = \frac{\omega}{\sqrt{2}}\left(\hat{x}_1 \sin\Omega t + \hat{x}_2 \cos\Omega t + \hat{x}_3\right) \ .$$

Note that $|\vec{\omega}|$ is constant. The direction of $\vec{\omega}$ traces a circular path about the mountain, starting at the north pole and getting as far south as the equator. Let's figure out its speed along the earth's surface. Let $\hat{\omega}$ be a unit vector in the direction of $\vec{\omega}$: $\hat{\omega} = \vec{\omega}/\omega$. Then the point at which the angular velocity vector intersects the earth's surface is $\mathbf{q} = R\hat{\omega}$. In our chosen coordinate system,

$$\mathbf{q} = \frac{R}{\sqrt{2}}\left(\hat{x}_1 \sin\Omega t + \hat{x}_2 \cos\Omega t + \hat{x}_3\right)$$

The speed of this point is

$$\begin{aligned} |\dot{\mathbf{q}}| &= \left|\frac{R\Omega}{\sqrt{2}}(\hat{x}_1 \cos\Omega t - \hat{x}_2 \sin\Omega t)\right| \\ &= \frac{R\Omega}{\sqrt{2}} = \frac{\Delta I \omega R}{2I_1} \end{aligned}$$

$\Delta I = mR^2$, and $I_1 = \left(\frac{2}{5}M + m\right)R^2 \approx \frac{2}{5}MR^2$. The earth's angular frequency is $\omega = 7.29 \times 10^{-5} s^{-1}$ and its radius is $R = 6.38 \times 10^6$ m, so

$$\begin{aligned} |\dot{\mathbf{q}}| &= \frac{5}{4}\frac{m}{M}R\omega = 5.81 \times 10^{-7}\frac{\text{m}}{\text{s}} \\ &= 1.14 \times 10^{-2}\frac{\text{miles}}{\text{year}} \end{aligned}$$

5.
Assume that the earth is a rigid solid ellipsoid of revolution, rotating about its symmetry axis $\hat{\mathbf{x}}_3$, and that it has $1 - (I_2/I_3) = -0.0033$ (actually the earth *bulges* at the equator, so that this quantity is really positive). Two equal mountains are placed opposite each other on the equator "at speed", so that ω is the same immediately afterward. What fraction of the earth's mass must each mountain have in order to render the earth's rotation barely unstable with respect to small deviations of $\hat{\omega}$ from the $\hat{\mathbf{x}}_3$ axis?

Solution:
Use a coordinate system with $\hat{\mathbf{x}}_3$ pointing towards the north pole. Assume the mountains are added to the earth's surface along the positive and negative $\hat{\mathbf{x}}_1$ axis. Then the inertia tensor is diagonal both before and after the addition of the mountains. The earth's inertia tensor has diagonal elements (I_2, I_2, I_3), and the mountains contribute $(0, 2mR^2, 2mR^2)$, so the total inertia tensor has diagonal elements $(I_2, I_2 + 2mR^2, I_3 + 2mR^2)$ along the three coordinate directions. Before the mountains are added, the moment about the $\hat{\mathbf{x}}_3$ axis is the smallest. We know that in general a rotation about one of the principal axes is unstable if the moment of inertia about the axis is in between the other two. So instability sets in when

$$I_3 + 2mR^2 > I_2$$

(The other condition, $I_3 + 2mR^2 < I_2 + 2mR^2$ is always satisfied, since $I_3 < I_2$.) Rearrange this equation to get

$$m > \frac{I_2 - I_3}{2R^2} = 0.0033\frac{I_3}{2R^2} = 0.0033 \times \frac{1}{5}M$$

So the condition is $m > 0.00066M$.

6.
Consider an asymmetric body (principal moments $I_3 > I_2 > I_1$) initially rotating with $\vec{\omega}$ very close to the $\hat{x}_3$ axis.
(a)
Show that the projection of $\vec{\omega}(t)$ on the $\hat{x}_1 - \hat{x}_2$ plane describes an *ellipse*.
Solution:
Euler's equations are

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 \ . \end{aligned}$$

Differentiating the $\dot{\omega}_1$ equation with respect to time yields:

$$\begin{aligned} \ddot{\omega}_1 &= \frac{I_2 - I_3}{I_1}(\dot{\omega}_2\omega_3 + \dot{\omega}_3\omega_2) \\ &= \frac{I_2 - I_3}{I_1}\left(\omega_3\frac{I_3 - I_1}{I_2}\omega_3\omega_1 + \omega_2\frac{I_1 - I_2}{I_3}\omega_1\omega_2\right) \\ &= \frac{I_2 - I_3}{I_1}\omega_1\left({\omega_3}^2\frac{I_3 - I_1}{I_2} - {\omega_2}^2\frac{I_2 - I_1}{I_3}\right) \end{aligned}$$

Since $I_3 - I_1 > I_2 - I_1$, $\frac{1}{I_2} > \frac{1}{I_3}$, and ${\omega_3}^2 \gg {\omega_2}^2$, the first term in parentheses dominates the second. Therefore

$$\ddot{\omega}_1 \simeq -\left(\frac{I_3}{I_1} - 1\right)\left(\frac{I_3}{I_2} - 1\right){\omega_3}^2\omega_1$$

By a similar argument, we have:

$$\ddot{\omega}_2 \simeq -\left(\frac{I_3}{I_1} - 1\right)\left(\frac{I_3}{I_2} - 1\right){\omega_3}^2\omega_2$$

Thus both ω_1 and ω_2 execute simple harmonic motion with the same angular frequency

$$\Omega \equiv \left(\left(\frac{I_3}{I_1} - 1\right)\left(\frac{I_3}{I_2} - 1\right)\right)^{\frac{1}{2}}\omega_3 \ .$$

This will trace out what is known as a *Lissajous figure* in the ω_1-ω_2 plane, which is an ellipse if the frequencies of the two oscillators are identical, as they are in this case.

(**b**)

Calculate the ratio of the major and minor axes of the ellipse.

Solution:

Let

$$\begin{aligned}\omega_1 &= Re\left(\tilde{\omega}_1 e^{i\Omega t}\right)\\ \omega_2 &= Re\left(\tilde{\omega}_2 e^{i\Omega t}\right) \ .\end{aligned}$$

We choose to solve the complex equation of which this is the real part. Substituting this into the $\dot{\omega}_2$ Euler equation yields

$$\begin{aligned}I_2\left(i\Omega\tilde{\omega}_2 e^{i\Omega t}\right) &= \omega_3(I_3 - I_1)\left(\tilde{\omega}_1 e^{i\Omega t}\right)\\ \frac{\tilde{\omega}_2}{\tilde{\omega}_1} &= -\frac{i\omega_3}{\Omega}\frac{I_3 - I_1}{I_2}\end{aligned}$$

Thus $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are $\frac{\pi}{2}$ out of phase, and the ellipse axes are along ω_1 and ω_2. Thus the ratio of major and minor axes is:

$$\begin{aligned}\frac{|\tilde{\omega}_1|}{|\tilde{\omega}_2|} &= \frac{\omega_3}{\Omega}\frac{I_3 - I_1}{I_2}\\ &= \left(\frac{I_3 - I_1}{I_3 - I_2}\frac{I_1}{I_2}\right)^{\frac{1}{2}}\end{aligned}$$

This is < 1 or > 1 depending on the details of the inertia tensor.

7.

Consider a heavy symmetrical top with one point fixed. Show that the *magnitude* of the top's angular momentum about the fixed point can be expressed as a function only of the constants of motion and the polar angle θ of the top's axis.

Solution:

In terms of the Euler angles θ, ϕ, and ψ, the ω in the body axes can be written as

$$\begin{aligned}\omega = \ &\hat{\mathbf{x}}_\mathbf{1}\left(\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi\right)\\ &+ \hat{\mathbf{x}}_\mathbf{2}\left(-\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi\right)\\ &+ \hat{\mathbf{x}}_\mathbf{3}\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\end{aligned}$$

and the inertia tensor as

$$\mathcal{I} = \begin{pmatrix} I & 0 & 0\\ 0 & I & 0\\ 0 & 0 & I_3\end{pmatrix} \ .$$

The angular momentum $\mathbf{L}$ is therefore

$$\begin{aligned}\mathbf{L} &= \mathcal{I}\omega\\ &= I\omega_1\hat{\mathbf{x}}_\mathbf{1} + I\omega_2\hat{\mathbf{x}}_\mathbf{2} + I_3\omega_3\hat{\mathbf{x}}_\mathbf{3}\\ L^2 &= I^2\left(\omega_1^2 + \omega_2^2\right) + I_3^2\omega_3^2\\ &= I^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + I_3^2\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2\end{aligned}$$

From the notes (eqs. 11.2 and 11.4) we have two conserved quantities, p_ψ and E:

$$\begin{aligned}p_\psi &= I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\\ E &= \frac{I}{2}\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{{p_\psi}^2}{2I_3} + mgh\cos\theta\end{aligned}$$

These can be rearranged to yield:

$$\begin{aligned}\dot{\phi}\cos\theta + \dot{\psi} &= \frac{p_\psi}{I_3}\\ \dot{\theta}^2 + \dot{\phi}^2\sin^2\theta &= \frac{2}{I}\left(E - \frac{{p_\psi}^2}{2I_3} - mgh\cos\theta\right)\end{aligned}$$

Inserting these quantities into the above expression for L^2, and simplifying, gives us:

$$L^2 = 2I\left(E - \frac{{p_\psi}^2}{2I_3} - mgh\cos\theta\right) + {p_\psi}^2$$

which is a function only of θ and constants of motion.

8.

Investigate the motion of the heavy symmetrical top with one point fixed for the case in which the axis of rotation is vertical (along $\hat{\mathbf{x}}_3$). Show that the motion is either stable or unstable depending on whether the quantity $4I_2Mhg/I_3^2\omega_3^2$ is less than or greater than unity. (If the top is set spinning in the stable configuration ("sleeping"), it will become unstable as friction gradually reduces the value of ω_3. This is a familiar childhood observation.)

Solution:

Eq. 11.4 in the notes tells us:

$$E = \frac{(p_\phi - p_\psi \cos\theta)^2}{2I\sin^2\theta} + \frac{I}{2}\dot{\theta}^2 + \frac{p_\psi^2}{2I_3} + Mgh\cos\theta$$

Using initial conditions $\theta = \phi = \psi = 0$, $\dot{\theta} = \dot{\phi} = 0$, and $\dot{\psi} = \omega_3$, along with equation 11.2 from the notes, gives us

$$\begin{aligned} p_\psi &= I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right) = I_3\omega_3 \\ p_\phi &= I\dot{\phi}\sin^2\theta + I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\cos\theta = I_3\omega_3 \end{aligned}$$

Inserting these into the above expression for E allows us to write $E = \frac{I}{2}\dot{\theta}^2 + V(\theta)$, where $V(\theta)$ is an effective potential and is defined as

$$V(\theta) = \frac{I_3^2\omega_3^2}{2I}\left(\frac{1-\cos\theta}{\sin\theta}\right)^2 + Mgh\cos\theta + \frac{1}{2}I_3\omega_3^2\ .$$

If we take derivatives of $V(\theta)$ with respect to θ, we find

$$\begin{aligned} \frac{dV}{d\theta} &= \frac{I_3^2\omega_3^2}{I}\frac{\sin\theta}{(1+\cos\theta)^2} - Mgh\sin\theta \\ \frac{d^2V}{d\theta^2} &= \frac{I_3^2\omega_3^2}{I}\frac{2+\cos\theta-\cos^2\theta}{(1+\cos\theta)^3} - Mgh\cos\theta \end{aligned}$$

At $\theta = 0$, $\frac{dV}{d\theta} = 0$, as expected since $\theta = 0$ is an equilibrium point. In order for it to be stable, we need $\frac{d^2V}{d\theta^2} > 0$ at $\theta = 0$:

$$\begin{aligned} \frac{d^2V}{d\theta^2}_{\theta=0} &= \frac{I_3^2\omega_3^2}{4I} - Mgh > 0 \\ &\frac{4IMgh}{I_3^2\omega_3^2} < 1\ . \end{aligned}$$

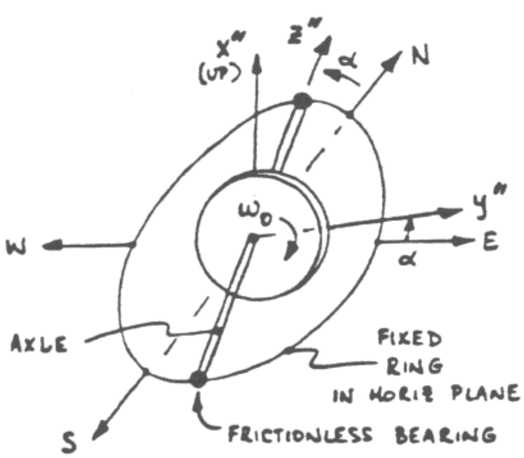
University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

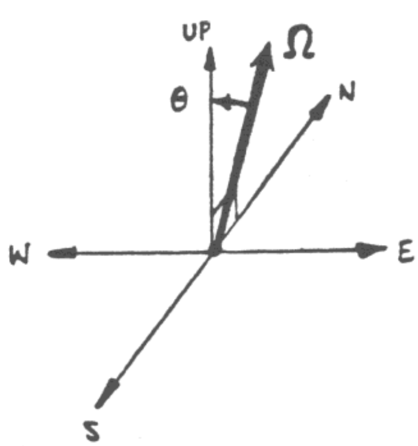
ASSIGNMENT 10

Reading:
105 Notes 12.1-12.4
Hand & Finch 9.1-9.6

1. and **2.** (double credit problem)
The *Foucault gyrocompass* is a gyroscope that eventually, taking advantage of frictional damping, points to true (not magnetic) north. Thus it is an essential guidance system component.

The gyrocompass may be modeled as a thin disk spinning with angular frequency ω_0 about its symmetry axis z''. This axis can move freely in the horizontal (North-South-East-West) plane only. As exhibited in the following diagrams, the z'' axis makes an angle $\alpha(t)$ with North. The gyrocompass is located at colatitude θ on an earth spinning with angular frequency Ω.





Assuming that $\omega_0 \gg \Omega$ and $\omega_0 \gg \dot{\alpha}$, prove that $\alpha(t)$ oscillates about $\alpha = 0$ provided that $\alpha \ll 1$. Find the angular frequency of oscillation. Note that friction in the bearings will eventually damp out this oscillation, enabling the gyrocompass to point to true north, as defined by the earth's axis of rotation.

You may find the following hints useful:

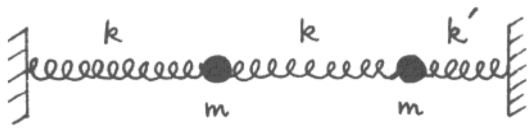
- Work the problem in the body ($''$) system. This system is obviously not the same as the fixed ($'$) system. It is also not the same as the unprimed system, which is the North-South-East-West system attached to the earth. Using Euler's equations would require knowing the torque from the bearings, evaluated in the body system. Since this torque is not known *a priori*, Euler's equations are not useful here.

- Write $\omega_{x''}$, $\omega_{y''}$, and $\omega_{z''}$ in terms of Ω, α, $\dot{\alpha}$, and θ.

- To get the relationship between the torque $\mathbf{N}'$ applied by the bearings and the angular momentum $\mathbf{L}''$, first write $\mathbf{N}' = d\mathbf{L}'/dt$ (taking advantage of the fact that the ($'$) system is inertial.) Then transform $\mathbf{L}'$ to the $''$ system.

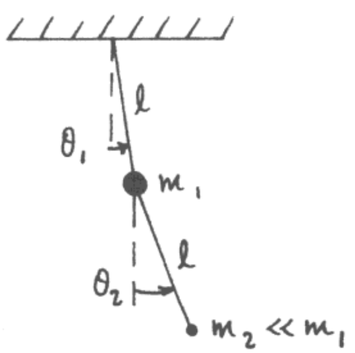
- When evaluating $\mathbf{L}$, remember to neglect terms that are smaller by a factor Ω/ω_0 than the leading terms.

3.
Consider a coupled oscillator with two equal masses m, each connected to fixed supports by springs with unequal spring constants k and k'. The two masses are connected to each other by a spring with spring constant k.



Find its two natural angular frequencies.

4.
Consider a double pendulum as exhibited in the following diagram. The two pendula are of equal lengths ℓ, but the lower mass $m_2 \ll m_1$. Choose θ_1 and θ_2, the angles between each string and the vertical, as generalized coordinates.

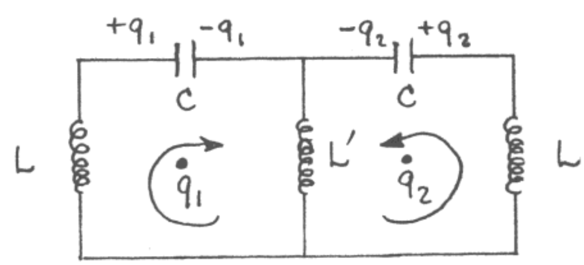


(**a**)
Find the natural angular frequencies of oscillation.
(**b**)
Calculate the interval $\mathcal{T}/2$ between times for which one or the other bob has minimum amplitude of oscillation. [*Hint:* This is $\pi/\Delta\omega$, where $\Delta\omega$ is the difference between the two natural angular frequencies.]

5.
Consider a linear triatomic molecule, as in the diagram below. A mass M is connected to two masses m, one on either side, by springs of equal spring constant k.

(**a**)
Find the three natural frequencies of the linear triatomic molecule.
(**b**)
One of these frequencies should be zero. To what motion does it correspond?

6.
In a series LC circuit, choose the charge q and its first derivative $\dot{q}$ as independent variables. Equate the "kinetic energy" T to $\frac{1}{2}L\dot{q}^2$ and the "potential energy" U to $\frac{1}{2}q^2/C$. Then Lagrange's equations produce the usual differential equation for the circuit.



In analogy with this approach, find the resonant frequencies of the above LC circuit. Do not rely on loop equations or any other circuit theory. Instead, write the analogous circuit Lagrangian and solve formally using coupled oscillator methods.

7.

Consider a thin homogeneous plate of mass M which lies in the $x_1 - x_2$ plane with its center at the origin. Let the length of the plate be $2A$ (in the x_2 direction) and let the width be $2B$ (in the x_1 direction). The plate is suspended from a fixed support by four springs of equal force constant k located at the four corners of the plate. The plate is free to oscillate, but with the constraint that its center must remain on the x_3 axis. Thus, there are 3 degrees of freedom: (1) vertical motion, with the center of the plate moving along the x_3 axis; (2) a tipping motion lengthwise, with the x_1 axis serving as an axis of rotation (choose an angle θ to describe this motion); and (3) a tipping motion sidewise, with the x_2 axis serving as an axis of rotation (choose an angle ϕ to describe this motion).

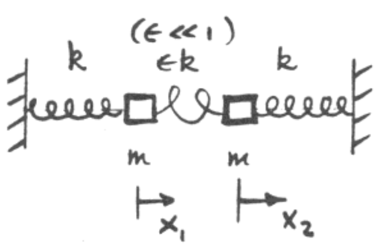
(**a**)
Assume only small oscillations and show that the secular equation has a double root and, hence, that the system is degenerate.

(**b**)
Discuss the normal modes of the system.

(**c**)
Show that the degeneracy can be removed by the addition to the plate of a thin bar of mass m and length $2A$ which is situated (at equilibrium) along the x_2 axis. Find the new eigenfrequencies of the system.

8.

Consider a pair of equal masses m connected to walls by equal springs with spring constant k. The two masses are connected to each other by a much weaker spring with spring constant ϵk, where $\epsilon \ll 1$. Choose x_1 and x_2, the displacements from equilibrium of the two masses, as the generalized coordinates.



For this system, write...

(**a**)
...the spring constant matrix $\mathcal{K}$ and the mass matrix $\mathcal{M}$

(**b**)
...the normal frequencies ω_1 and ω_2

(**c**)
...the normal mode vectors $\tilde{a}_1$ and $\tilde{a}_2$ (corresponding to ω_1 and ω_2), each expressed as a linear combination of x_1 and x_2

(**d**)
...the 2×2 matrix $\mathcal{A}$ which reduces $\mathcal{M}$ to the unit matrix via the congruence transformation

$$\mathcal{I} = \mathcal{A}^t \mathcal{M} \mathcal{A}\,,$$

where $\mathcal{I}$ is the identity matrix

(**e**)
...the normal coordinates Q_1 and Q_2 (corresponding to ω_1 and ω_2), each expressed as a linear combination of x_1 and x_2.

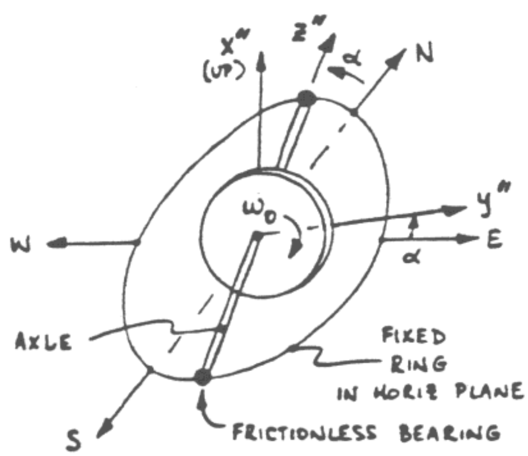
University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

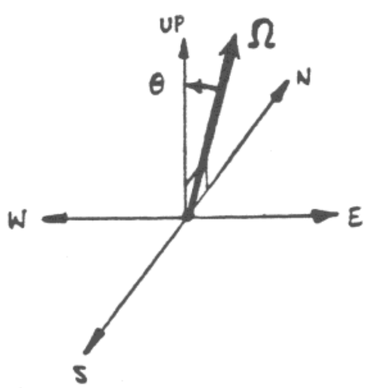
SOLUTION TO PROBLEM SET 10
Solutions by T. Bunn

Reading:
105 Notes 12.1-12.4
Hand & Finch 9.1-9.6

1. and **2.** (double credit problem)
The *Foucault gyrocompass* is a gyroscope that eventually, taking advantage of frictional damping, points to true (not magnetic) north. Thus it is an essential guidance system component.

The gyrocompass may be modeled as a thin disk spinning with angular frequency ω_0 about its symmetry axis z''. This axis can move freely in the horizontal (North-South-East-West) plane only. As exhibited in the following diagrams, the z'' axis makes an angle $\alpha(t)$ with North. The gyrocompass is located at colatitude θ on an earth spinning with angular frequency Ω.





Assuming that $\omega_0 \gg \Omega$ and $\omega_0 \gg \dot{\alpha}$, prove that $\alpha(t)$ oscillates about $\alpha = 0$ provided that $\alpha \ll 1$. Find the angular frequency of oscillation. Note that friction in the bearings will eventually damp out this oscillation, enabling the gyrocompass to point to true north, as defined by the earth's axis of rotation.

You may find the following hints useful:

- Work the problem in the body ($''$) system. This system is obviously not the same as the fixed ($'$) system. It is also not the same as the unprimed system, which is the North-South-East-West system attached to the earth. Using Euler's equations would require knowing the torque from the bearings, evaluated in the body system. Since this torque is not known *a priori*, Euler's equations are not useful here.
- Write $\omega_{x''}$, $\omega_{y''}$, and $\omega_{z''}$ in terms of Ω, α, $\dot{\alpha}$, and θ.
- To get the relationship between the torque $\mathbf{N}'$ applied by the bearings and the angular momentum $\mathbf{L}''$, first write $\mathbf{N}' = d\mathbf{L}'/dt$ (taking advantage of the fact that the ($'$) system is inertial.) Then transform $\mathbf{L}'$ to the $''$ system.
- When evaluating $\mathbf{L}$, remember to neglect terms that are smaller by a factor Ω/ω_0 than the leading terms.

Solution:
One thing that is certainly going to prove useful is the angular velocity vector of the gyroscope. It is the sum of three parts: the rotation $\vec{\omega}_0$ about the gyroscope axis; the rate of change of the gyroscope axis direction, which is equal to $\dot{\alpha}$, and which points in the $\hat{x}''$ direction; and the rotation of the earth, $\vec{\Omega}$. In the $''$ coordinate

system, these all add up to

$$\vec{\omega} = \hat{x}''(\dot{\alpha} + \Omega\cos\theta) + \hat{y}''\Omega\sin\theta\sin\alpha + \hat{z}''(\omega_0 + \Omega\sin\theta\cos\alpha)\ .$$

From this we can find the angular momentum vector in the $''$ system, since the inertia tensor I is diagonal in this system: $\vec{L} = I_1\omega_{x''}\hat{x}'' + I_1\omega_{y''}\hat{y}'' + I_3\omega_{z''}\hat{z}''$.

We want to apply the rule $\vec{N} = \dot{\vec{L}}$, but we can't, because the $''$ system is not inertial. We can, however, say this:

$$\vec{N} = \left(\frac{d\vec{L}}{dt}\right)_{\text{inertial}} = \left(\frac{d\vec{L}}{dt}\right)_{\text{rotating}} + \vec{\omega}^* \times \vec{L}$$

where $\vec{\omega}^*$ is the angular velocity vector of the rotating coordinate system with respect to the inertial coordinate system. Let's consider the $''$ system as our rotating system. Then $\vec{\omega}^*$ is the sum of two components: the rotation of the earth $\vec{\Omega}$, and the rotation of the gyroscope axis, $\dot{\alpha}\hat{x}''$. The components of $\vec{\omega}^*$ in the $''$ system are

$$\vec{\omega}^* = \hat{x}''(\dot{\alpha} + \Omega\cos\theta) + \hat{y}''\Omega\sin\theta\sin\alpha + \hat{z}''\Omega\sin\theta\cos\alpha\ .$$

Now, we're interested in the motion of the gyroscope about the x'' axis, so let's write down the x'' component of our torque equation. There is no torque in this direction (because the bearing is frictionless), so

$$N_{x''} = 0 = \left(\frac{d\vec{L}''}{dt}\right)_{x''} + \left(\vec{\omega}^* \times \vec{L}\right)_{x''}$$

(Note that $(d\vec{L}''/dt)_{x''}$ means the x'' component of the time derivative of $\vec{L}$ as seen in the $''$ system.) We know the components of all of these vectors in the $''$ system, so we can write this expression explicitly:

$$0 = I_1\ddot{\alpha} + I_3\Omega\sin\theta\sin\alpha(\omega_0 + \Omega\sin\theta\cos\alpha) - I_1\Omega^2\sin^2\theta\sin\alpha\cos\alpha\ .$$

To simplify this equation, note that Ω is much smaller than any other frequency in the problem. So let's drop all terms higher than first order in Ω.

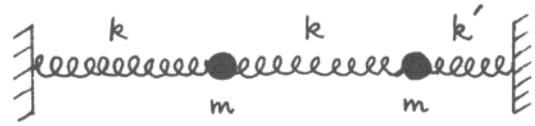
$$I_1\ddot{\alpha} + I_3\omega_0\Omega\sin\theta\sin\alpha = 0$$

Approximate $\sin\alpha \approx \alpha$, and you get the harmonic oscillator equation. α thus oscillates about 0 (true north) with frequency γ, given by

$$\gamma^2 = \frac{I_3}{I_1}\omega_0\,\Omega\sin\theta.$$

3.

Consider a coupled oscillator with two equal masses m, each connected to fixed supports by springs with unequal spring constants k and k'. The two masses are connected to each other by a spring with spring constant k.



Find its two natural angular frequencies.

Solution:

Let x_1 and x_2 be the displacements of the two masses from their equilibrium positions. Then the forces on the two masses are $F_1 = -kx_1 - k(x_1 - x_2)$, and $F_2 = -k'x_2 - k(x_2 - x_1)$. So the equations of motion are

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$
$$m\ddot{x}_2 - kx_1 + (k + k')x_2 = 0$$

Guess that the solutions are periodic: $x_j = A_j e^{i\omega t}$. Then we get a pair of linear equations for A_1 and A_2:

$$\left(2k - m\omega^2\right)A_1 - kA_2 = 0$$
$$-kA_1 + \left(k' + k - m\omega^2\right)A_2 = 0$$

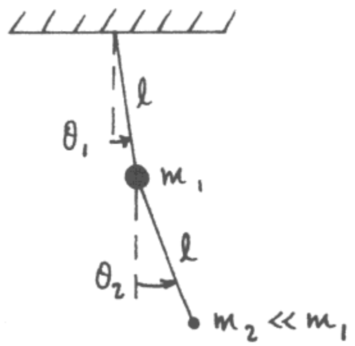
These equations only have nontrivial solutions if the determinant of the coefficients is zero:

$$0 = \begin{vmatrix} 2k - m\omega^2 & -k \\ -k & k + k' - m\omega^2 \end{vmatrix} = m^2\omega^4 - m(3k + k')\omega^2 + k^2 + 2kk'$$

The solutions to this quadratic equation for ω^2 are the two natural frequencies:

$$\omega^2 = \frac{3k + k' \pm \sqrt{5k^2 - 2kk' + k'^2}}{2m}$$

4.
Consider a double pendulum as exhibited in the following diagram. The two pendula are of equal lengths ℓ, but the lower mass $m_2 \ll m_1$. Choose θ_1 and θ_2, the angles between each string and the vertical, as generalized coordinates.



(a)
Find the natural angular frequencies of oscillation.
Solution:
Let's write a Lagrangian. The kinetic energy of mass 1 is easy: $T_1 = \frac{1}{2}m_1 l^2 \dot{\theta}_1^2$. The potential is easy too: $V = -m_1 g l \cos\theta_1 - m_2 g l(\cos\theta_1 + \cos\theta_2)$. Now what about T_2? Well, the position of mass 2 has x, y coordinates $\vec{r}_2 = (l\sin\theta_1 + l\sin\theta_2, -l\cos\theta_1 - l\cos\theta_2)$. Take the time derivative and square to get v_2^2. Then you find that

$$T_2 = \tfrac{1}{2}m_2 l^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right)$$

Putting it all together, we get

$$\begin{aligned}\mathcal{L} = \tfrac{1}{2}l^2 &\big((m_1 + m_2)\dot{\theta}_1^2 + m_2\dot{\theta}_2^2 \\ &+ 2m_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\big) \\ &+ (m_1 + m_2)gl\cos\theta_1 + m_2 gl\cos\theta_2 \ .\end{aligned}$$

The two Euler-Lagrange equations are

$$\begin{aligned}0 &= (m_1 + m_2)l^2\ddot{\theta}_1 + m_2 l^2\ddot{\theta}_2\cos(\theta_1 - \theta_2) \\ &\quad + m_2 l^2\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + (m_1 + m_2)gl\sin\theta_1 \\ 0 &= m_2 l^2\ddot{\theta}_2 + m_2 l^2\ddot{\theta}_1\cos(\theta_1 - \theta_2) \\ &\quad - m_2 l^2\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + m_2 gl\sin\theta_2 \ .\end{aligned}$$

We're clearly not going to get anywhere without making some approximations: Start with the small-angle approximation: Set $\sin\theta = \theta$, $\cos\theta = 1$, and drop all terms with more than one power of θ:

$$\begin{aligned}(m_1 + m_2)l^2\ddot{\theta}_1 + m_2 l^2\ddot{\theta}_2 + (m_1 + m_2)gl\theta_1 &= 0 \\ m_2 l^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 gl\theta_2 &= 0\end{aligned}$$

That's better. Now we solve these differential equations in the usual way: by guessing the answer. Assume solutions of the form

$$\begin{aligned}\theta_1 &= A_1 e^{i\omega t} \\ \theta_2 &= A_2 e^{i\omega t}\end{aligned}$$

Substitute these expressions for θ_1 and θ_2, and you get

$$\begin{aligned}(m_1 + m_2)l(g - l\omega^2)A_1 - m_2 l^2\omega^2 A_2 &= 0 \\ -m_2 l^2\omega^2 A_1 + m_2 l(g - l\omega^2)A_2 &= 0\end{aligned} \qquad (1)$$

There's no nontrivial solution unless the determinant of the coefficients is zero:

$$\begin{vmatrix}(m_1 + m_2)l(g - l\omega^2) & -m_2 l^2\omega^2 \\ -m_2 l^2\omega^2 & m_2 l(g - l\omega^2)\end{vmatrix} = 0$$

Some notation: Define $\omega_0^2 = g/l$, and $\epsilon = m_2/m_1$. Then computing the determinant and canceling some terms, we get

$$\epsilon(1 + \epsilon)\left(\omega_0^2 - \omega^2\right)^2 - \epsilon^2\omega^4 = 0$$

This equation has two solutions for ω^2, which we'll call ω_+ and ω_-.

$$\omega_\pm^2 = \omega_0^2\left(1 + \epsilon \pm \sqrt{\epsilon(1 + \epsilon)}\right)$$

These are the "natural frequencies" of this system.
(b)
Calculate the interval $\mathcal{T}/2$ between times for which one or the other bob has minimum amplitude of oscillation. [*Hint:* This is $\pi/\Delta\omega$, where

$\Delta\omega$ is the difference between the two natural angular frequencies.]

Solution:

Rather than accept the hint, which greatly simplifies this part of the problem, why don't we take this opportunity to work out the motion completely. Then the interval $\mathcal{T}/2$ will fall out. First we figure out the amplitudes A_1 and A_2 that go with $\omega_\pm$: From equation (1) above, we get

$$\frac{A_1^\pm}{A_2^\pm} = \frac{\epsilon(\omega_0^2 - \omega_\pm^2)}{\epsilon\omega_\pm^2} = \mp\sqrt{\frac{\epsilon}{1+\epsilon}}$$

(This comes from the second equation in (1), although the first one would have worked just as well. We've skipped some steps in simplifying it.) So far we haven't made the approximation $m_2 \ll m_1$ (*i.e.*, $\epsilon \ll 1$). Let's use it now to say $A_1^\pm/A_2^\pm \approx \mp\sqrt{\epsilon}$. Only the ratio of A_1 to A_2 is determined by the equations of motion, so we can pick the overall magnitude any way we want. Let's say the following:

$$A_1^\pm = \mp\sqrt{\epsilon} \qquad\qquad A_2^\pm = 1$$

The most general solution to the equations of motion will be a linear combination of the + and − solutions:

$$\begin{pmatrix}\theta_1(t)\\ \theta_2(t)\end{pmatrix} = \mathrm{Re}\left(c_+\begin{pmatrix}A_1^+\\ A_2^+\end{pmatrix}e^{i\omega_+ t}\right) + \mathrm{Re}\left(c_-\begin{pmatrix}A_1^-\\ A_2^-\end{pmatrix}e^{i\omega_- t}\right) .$$

Let's say that at $t = 0$, $\dot\theta_1 = \dot\theta_2 = 0$, and $\theta_1 = \theta_2 = \theta_0$. Since the initial velocities are zero, we can take c_+ and c_- to be real, and replace the complex exponentials by cosines. Then our initial conditions say that

$$\begin{pmatrix}(c_- - c_+)\sqrt{\epsilon}\\ c_- + c_+\end{pmatrix} = \begin{pmatrix}\theta_0\\ \theta_0\end{pmatrix}$$

so $c_+ = \frac{1}{2}\theta_0(1-\epsilon^{-1/2})$ and $c_- = \frac{1}{2}\theta_0(1+\epsilon^{-1/2})$. Putting it all together, we get

$$\begin{aligned}\theta_1(t) &= \tfrac{1}{2}\theta_0\Big((1-\epsilon^{1/2})\cos\omega_+ t\\ &\qquad + (1+\epsilon^{1/2})\cos\omega_- t\Big)\\ \theta_2(t) &= \tfrac{1}{2}\theta_0\Big((1-\epsilon^{-1/2})\cos\omega_+ t\\ &\qquad + (1+\epsilon^{-1/2})\cos\omega_- t\Big) .\end{aligned}$$

Now let's return to our original goal of finding the time interval $\mathcal{T}/2$ between maximum and minimum amplitudes of oscillation for one bob. Let's concentrate on θ_1. At $t = 0$, the two terms in the expression for θ_1 are in phase with each other. After a certain time, since $\omega_+ \neq \omega_-$, the two terms will be 180° out of phase, and the amplitude will be minimized. This happens after a time $\mathcal{T}/2 = \pi/(\omega_+ - \omega_-)$. Making our usual small-ϵ argument, the frequency difference is

$$\begin{aligned}\frac{\omega_+ - \omega_-}{\omega_0} &= \sqrt{1+\epsilon+\sqrt{\epsilon(1+\epsilon)}}\\ &\quad - \sqrt{1+\epsilon-\sqrt{\epsilon(1+\epsilon)}}\\ &\approx 1 + \tfrac{1}{2}\epsilon + \tfrac{1}{2}\sqrt{\epsilon(1+\epsilon)}\\ &\quad - 1 - \tfrac{1}{2}\epsilon + \tfrac{1}{2}\sqrt{\epsilon(1+\epsilon)}\\ &= \sqrt{\epsilon(1+\epsilon)}\\ &\approx \sqrt{\epsilon} .\end{aligned}$$

So the time between maximum and minimum oscillation of mass 1 is $\mathcal{T}/2 \approx \pi/\omega_0\sqrt{\epsilon}$.

5.

Consider a linear triatomic molecule, as in the diagram below. A mass M is connected to two masses m, one on either side, by springs of equal spring constant k.

(a)

Find the three natural frequencies of the linear triatomic molecule.

Solution:

If x_1, x_2, x_3 are the displacements of the three atoms from their equilibrium positions, then the equations of motion are

$$\begin{aligned} m\ddot{x}_1 + k(x_1 - x_2) &= 0\\ M\ddot{x}_2 + k(2x_2 - x_1 - x_3) &= 0\\ m\ddot{x}_3 + k(x_3 - x_2) &= 0\end{aligned}$$

As usual, guess that the solutions have time dependence $e^{i\omega t}$. Then there are only solutions if the secular determinant is zero:

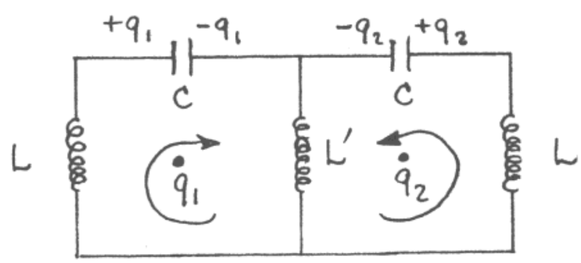
$$0 = \begin{vmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - M\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{vmatrix}$$
$$= -\omega^2 \big(Mm^2\omega^4 - 2k(Mm + m^2)\omega^2 + k^2(M + 2m)\big) \ .$$

This cubic equation for ω^2 has three solutions:

$$\omega^2 = 0 \quad \omega^2 = \frac{k}{m} \quad \omega^2 = \frac{k}{Mm}(M + 2m)$$

(**b**)
One of these frequencies should be zero. To what motion does it correspond?
Solution:
The zero-frequency solution corresponds to uniform translation of the molecule, with no stretching of the springs at all.

6.
In a series LC circuit, choose the charge q and its first derivative $\dot{q}$ as independent variables. Equate the "kinetic energy" T to $\frac{1}{2}L\dot{q}^2$ and the "potential energy" U to $\frac{1}{2}q^2/C$. Then Lagrange's equations produce the usual differential equation for the circuit.



In analogy with this approach, find the resonant frequencies of the above LC circuit. Do not rely on loop equations or any other circuit theory. Instead, write the analogous circuit Lagrangian and solve formally using coupled oscillator methods.
Solution:
The Lagrangian for this system is

$$\mathcal{L} = \tfrac{1}{2}L\left(\dot{q}_1^2 + \dot{q}_2^2\right) + \tfrac{1}{2}L'\left(\dot{q}_1 + \dot{q}_2\right)^2 - \frac{1}{2C}\left(q_1^2 + q_2^2\right)$$

which gives equations of motion

$$(L + L')\ddot{q}_1 + L'\ddot{q}_2 + q_1/C = 0$$
$$L'\ddot{q}_1 + (L + L')\ddot{q}_2 + q_2/C = 0$$

The secular determinant is

$$0 = \begin{vmatrix} \frac{1}{C} - (L + L')\omega^2 & -L'\omega^2 \\ -L'\omega^2 & \frac{1}{C} - (L + L')\omega^2 \end{vmatrix}$$
$$= \left((L + L')\omega^2 - \frac{1}{C}\right)^2 - L'^2\omega^4 \ .$$

The solutions are

$$\omega^2 = \frac{1}{LC} \qquad\qquad \omega^2 = \frac{1}{(L + 2L')C} \ .$$

7.
Consider a thin homogeneous plate of mass M which lies in the $x_1 - x_2$ plane with its center at the origin. Let the length of the plate be $2A$ (in the x_2 direction) and let the width be $2B$ (in the x_1 direction). The plate is suspended from a fixed support by four springs of equal force constant k located at the four corners of the plate. The plate is free to oscillate, but with the constraint that its center must remain on the x_3 axis. Thus, there are 3 degrees of freedom: (1) vertical motion, with the center of the plate moving along the x_3 axis; (2) a tipping motion lengthwise, with the x_1 axis serving as an axis of rotation (choose an angle θ to describe this motion); and (3) a tipping motion sidewise, with the x_2 axis serving as an axis of rotation (choose an angle ϕ to describe this motion).
(**a**)
Assume only small oscillations and show that the secular equation has a double root and, hence, that the system is degenerate.
Solution:
Let's choose generalized coordinates as follows: Let z be the height of the center of mass, and θ and ϕ be angles of rotation about the x_1 and x_3 axes. Then the kinetic energy is

$$T = \tfrac{1}{2}M\dot{z}^2 + \tfrac{1}{2}I_1\dot{\theta}^2 + \tfrac{1}{2}I_2\dot{\phi}^2$$

The potential energy stored in each spring is just $\frac{1}{2}k$ times the height^2 of the corresponding corner of the slab. The height of the corner in the first quadrant of the x_1-x_2 plane is $z + A\theta - B\phi$ for small θ and ϕ. There are similar expressions with different $+$ and $-$ signs for the other three heights, giving

$$\begin{aligned} V &= \tfrac{1}{2}k\big((A\theta - B\phi + z)^2 + (A\theta + B\phi + z)^2 \\ &\quad + (-A\theta + B\phi + z)^2 + (-A\theta - B\phi + z)^2\big) \\ &= 2k\left(A^2\theta^2 + B^2\phi^2 + z^2\right) \ . \end{aligned}$$

Now set $\mathcal{L} = T - V$ and get the Euler-Lagrange equations

$$\begin{aligned} M\ddot{z} + 4kz &= 0 \\ I_1\ddot{\theta} + 4A^2k\theta &= 0 \\ I_2\ddot{\phi} + 4B^2k\phi &= 0 \end{aligned}$$

The secular determinant is pretty easy:

$$\begin{aligned} 0 &= \begin{vmatrix} 4k - M\omega^2 & 0 & 0 \\ 0 & 4kA^2 - I_1\omega^2 & 0 \\ 0 & 0 & 4kB^2 - I_2\omega^2 \end{vmatrix} \\ &= (4k - M\omega^2)(4kA^2 - I_1\omega^2)(4kB^2 - I_2\omega^2) \ . \end{aligned}$$

The natural frequencies are

$$\omega_1^2 = 4k/M \quad \omega_2^2 = 4kA^2/I_1 \quad \omega_3^2 = 4kB^2/I_2$$

The moments of inertia are $I_1 = \frac{1}{3}MA^2$ and $I_2 = \frac{1}{3}MB^2$, so the last two frequencies are the same: $\omega_2^2 = \omega_3^2 = 12k/M$.

(b)

Discuss the normal modes of the system.

Solution:

The normal modes associated with these three roots of the secular equation are as follows: (1) $\theta = \phi = 0$, $z \propto \cos\omega t$. (Vertical motion; no twisting.) (2) $\phi = z = 0$, $\theta \propto \cos\omega t$. (Rotation about the x_1 axis.) (3) $\theta = z = 0$, $\phi \propto \cos\omega t$. (Rotation about the x_2 axis.) Of course, since modes 2 and 3 are degenerate (*i.e.*, have the same frequency), any linear combination of them could also be chosen as a normal mode.

(c)

Show that the degeneracy can be removed by the addition to the plate of a thin bar of mass m and length $2A$ which is situated (at equilibrium) along the x_2 axis. Find the new eigenfrequencies of the system.

Solution:

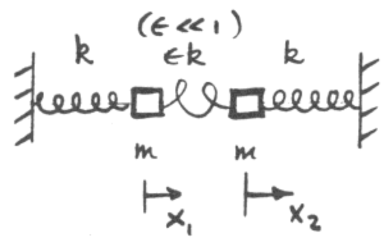
If we add a bar of mass m along the x_2 axis, then I_2 is unchanged, while $I_1 = \frac{1}{3}(M+m)A^2$. The three natural frequencies are

$$\begin{aligned} \omega_1^2 &= \frac{4k}{M+m} \\ \omega_2^2 &= \frac{4kA^2}{I_1} = \frac{12k}{M+m} \\ \omega_3^2 &= \frac{4kB^2}{I_2} = \frac{12k}{M} \end{aligned}$$

Since $I_1 \neq I_2$, there is no degeneracy.

8.

Consider a pair of equal masses m connected to walls by equal springs with spring constant k. The two masses are connected to each other by a much weaker spring with spring constant ϵk, where $\epsilon \ll 1$. Choose x_1 and x_2, the displacements from equilibrium of the two masses, as the generalized coordinates.



For this system, write...

(a)

...the spring constant matrix $\mathcal{K}$ and the mass matrix $\mathcal{M}$

Solution:

The Lagrangian for this system can be written as:

$$\begin{aligned} \mathcal{L} &= \left(\frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2\right) \\ &\quad - \left(\frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}\epsilon k(x_1 - x_2)^2\right) \\ &= \frac{1}{2}\left(m\dot{x}_1^2 + m\dot{x}_2^2\right) \\ &\quad - \frac{1}{2}\left(k(1+\epsilon)x_1^2 k(1+\epsilon)x_2^2 - 2\epsilon k x_1 x_2\right) \\ &= \frac{1}{2}\dot{\mathbf{x}} \cdot \mathcal{M}\dot{\mathbf{x}} - \frac{1}{2}\mathbf{x} \cdot \mathcal{K}\mathbf{x} \end{aligned}$$

where

$$\mathcal{M} = m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\mathcal{K} = k \begin{pmatrix} 1+\epsilon & -\epsilon \\ -\epsilon & 1+\epsilon \end{pmatrix}$$

(**b**)
...the normal frequencies ω_1 and ω_2
Solution:
Normal frequencies are given by:

$$\det\left(\mathcal{K} - \omega^2 \mathcal{M}\right) = 0$$
$$\begin{vmatrix} k(1+\epsilon) - m\omega^2 & -\epsilon k \\ -\epsilon k & k(1+\epsilon) - m\omega^2 \end{vmatrix} = 0$$

which yields

$$\omega_1^2 = \frac{k}{m}$$
$$\omega_2^2 = \frac{k + 2\epsilon k}{m}$$

(**c**)
...the normal mode vectors $\tilde{a}_1$ and $\tilde{a}_2$ (corresponding to ω_1 and ω_2), each expressed as a linear combination of x_1 and x_2
Solution:
The normal mode vectors $\vec{a}_1$ and $\vec{a}_2$ are determined by the conditions

$$\left(\mathcal{K} - \omega_i^2 \mathcal{M}\right) \vec{a}_i = 0$$
$$\vec{a}_i \cdot \mathcal{M} \vec{a}_i = 1$$

applied using each normal mode frequency in turn. This yields:

$$\vec{a}_1 = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\vec{a}_2 = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(**d**)
...the 2×2 matrix $\mathcal{A}$ which reduces $\mathcal{M}$ to the unit matrix via the congruence transformation

$$\mathcal{I} = \mathcal{A}^t \mathcal{M} \mathcal{A}\,,$$

where $\mathcal{I}$ is the identity matrix
Solution:
From Eq. 12.13 in the notes, $\mathcal{A}$ is the matrix of normal mode vectors:

$$\mathcal{A} = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(**e**)
...the normal coordinates Q_1 and Q_2 (corresponding to ω_1 and ω_2), each expressed as a linear combination of x_1 and x_2.
Solution:
From Eq. 12.15 in the notes:

$$\vec{Q} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \mathcal{A}^t \mathcal{M} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$= \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$= \sqrt{\frac{m}{2}} \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

So $Q_1 = \sqrt{\frac{m}{2}}(x_1 + x_2)$ and $Q_2 = \sqrt{\frac{m}{2}}(x_1 - x_2)$.

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ASSIGNMENT 11

Reading:
105 Notes 14.1-14.5
Hand & Finch 2.9, 9.7

1.
Discuss the motion of a continuous string (tension τ, mass per unit length μ) with fixed endpoints $y = 0$ at $x = 0$ and $x = L$, when the initial conditions are

$$y(x,0) = A\sin\frac{3\pi x}{L}$$
$$\dot{y}(x,0) = 0 \ .$$

Resolve the solution into normal modes.

2.
Discuss the motion of a continuous string (tension τ, mass per unit length μ) with fixed endpoints $y = 0$ at $x = 0$ and $x = L$, when (in a certain set of units) the initial conditions are

$$y(x,0) = 4\,\frac{x(L-x)}{L^2}$$
$$\dot{y}(x,0) = 0 \ .$$

Find the characteristic frequencies and calculate the amplitude of the n^{th} mode.

3.
Solve for the motion $y(x,t)$ of a continuous string (tension τ, mass per unit length μ) with fixed endpoints $y = 0$ at $x = 0$ and $x = L$, when the initial conditions are

$$y(x,0) = A\sin\frac{\pi x}{L}$$
$$\dot{y}(x,0) = V\sin\frac{5\pi x}{L} \ ,$$

where A and V are constants.

4.
A continuous string (tension τ, mass per unit length μ) is attached to fixed supports *infinitely far away*. At $t = 0$ the string satisfies initial conditions

$$y(x,0) = 0$$
$$\frac{\partial y}{\partial t}(x,0) = \alpha\,\delta(x) \ ,$$

where $\delta(x)$ is a Dirac delta function and α is a constant that can be made arbitrarily infinitesimal, so that the string's slope remains small enough for the usual wave equation to apply. This initial condition is appropriate to the string having been struck at $(x = 0, t = 0)$ with a sharp object.

Compute $y(x,t)$ for $t > 0$.

5.
Show that if ψ and ψ^* are taken as two *independent* field variables, the Lagrangian density

$$\mathcal{L}' = \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi + V\psi^*\psi + \frac{\hbar}{2i}(\psi^*\dot{\psi} - \psi\dot{\psi}^*)$$

(where $\dot{}$ means $\partial/\partial t$ in this context) leads to the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

and its complex conjugate.

6.
Consider a membrane stretched between *fixed* supports at $x = 0$, $x = L$, $y = 0$, and $y = L$. *Per unit area*, its kinetic and potential energies are

$$T' = \tfrac{1}{2}\sigma\Big(\frac{\partial z}{\partial t}\Big)^2$$
$$U' = \tfrac{1}{2}\beta\Big(\Big(\frac{\partial z}{\partial x}\Big)^2 + \Big(\frac{\partial z}{\partial y}\Big)^2\Big) \ ,$$

where σ is the membrane's mass per unit area, β is a constant that is inversely proportional to its elasticity, and z is its (normal) displacement.

Apply the Euler-Lagrange equations to obtain a partial differential equation for $z(x, y, t)$. Using a trial solution

$$z(x, y, t) = X(x)\, Y(y)\, T(t) \ ,$$

find the angular frequencies of vibration for the *five* lowest-frequency normal modes of oscillation.

7. and **8.** (double problem)
The Lagrangian density (per unit volume) for a charge density $\rho(\mathbf{r}, t)$ and current density $\mathbf{j}(\mathbf{r}, t)$ in the presence of an electromagnetic field $\mathbf{E}(\mathbf{r}, t)$, $\mathbf{B}(\mathbf{r}, t)$ is

$$\mathcal{L}' = \frac{E^2 - B^2}{8\pi} - \rho\phi + \frac{1}{c}\mathbf{j} \cdot \mathbf{A} \ .$$

The first term is the Lagrangian density corresponding to the self-energy of the free field, and the latter terms represent the interaction between fields and charges. The self-energy of the individual (point) charges is infinity in classical theory and is omitted. In the above, $\mathbf{A}$ is the *vector potential* defined by

$$\mathbf{B} = \nabla \times \mathbf{A}$$
$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

(Gaussian units are used throughout this problem). If you are familiar with relativistic transformations of electromagnetic fields, you may notice that the above Lagrangian density is *Lorentz invariant*, although not manifestly so.

The homogeneous (charge and current independent) Maxwell equations follow directly from the equations relating $\mathbf{E}$ and $\mathbf{B}$ to the potentials. To complete the picture, using ϕ and the three components of $\mathbf{A}$ as four generalized (field) coordinates, apply the Euler-Lagrange equations to $\mathcal{L}'$ to obtain the two inhomogeneous Maxwell equations

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$
$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j} \ .$$

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO PROBLEM SET 11

Solutions by T. Bunn and J. Barber

Reading:
105 Notes 14.1-14.5
Hand & Finch 2.9, 9.7

1.
Discuss the motion of a continuous string (tension τ, mass per unit length μ) with fixed endpoints $y = 0$ at $x = 0$ and $x = L$, when the initial conditions are

$$y(x, 0) = A \sin \frac{3\pi x}{L}$$
$$\dot{y}(x, 0) = 0\ .$$

Resolve the solution into normal modes.
Solution:
A general solution to this problem can be written as:

$$y(x, t) = \sum_{n=1}^{\infty} (A_n \sin \omega_n t + B_n \cos \omega_n t) \sin \frac{n\pi x}{L}$$

where $\omega_n = n\omega_1$ and $\omega_1 = \frac{\pi}{L}\sqrt{\frac{\tau}{\mu}}$. From our initial conditions, we have:

$$\begin{aligned} y(x, 0) &= \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \\ &= A \sin \frac{3\pi x}{L} \end{aligned}$$

Thus, by inspection, $B_3 = A$, and all the other B_n are zero. We also have the initial condition for velocity:

$$\begin{aligned} \dot{y}(x, 0) &= \sum_{n=1}^{\infty} \omega_n A_n \sin \frac{n\pi x}{L} \\ &= 0 \end{aligned}$$

Thus all the A_n are zero. So the full solution is:

$$y(x, t) = A \cos \omega_3 t \sin \frac{3\pi x}{L}$$

where $\omega_3 = \frac{3\pi}{L}\sqrt{\frac{\tau}{\mu}}$.

2.
Discuss the motion of a continuous string (tension τ, mass per unit length μ) with fixed endpoints $y = 0$ at $x = 0$ and $x = L$, when (in a certain set of units) the initial conditions are

$$y(x, 0) = 4\, \frac{x(L - x)}{L^2}$$
$$\dot{y}(x, 0) = 0\ .$$

Find the characteristic frequencies and calculate the amplitude of the n^{th} mode.
Solution:
Again using the expansion

$$y(x, t) = \sum_{n=1}^{\infty} (A_n \sin \omega_n t + B_n \cos \omega_n t) \sin \frac{n\pi x}{L}\ ,$$

we fit the initial conditions:

$$\begin{aligned} \dot{y}(x, 0) &= \sum_{n=1}^{\infty} \omega_n A_n \sin \frac{n\pi x}{L} \\ &= 0 \end{aligned}$$

Thus all the A_n are zero.

$$\begin{aligned} y(x, 0) &= \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \\ &= \frac{4x(L - x)}{L^2} \end{aligned}$$

where

$$B_n = \frac{2}{L} \int_0^L \frac{4x(L - x)}{L^2} \sin \frac{n\pi x}{L}\, dx$$

Here are two useful integrals:

$$\int_0^L x \sin\left(\frac{n\pi x}{L}\right)\,dx = (-1)^{n+1}\frac{L^2}{n\pi}$$

$$\int_0^L x^2 \sin\left(\frac{n\pi x}{L}\right)\,dx = \begin{cases} \frac{L^3}{n\pi} - \frac{4L^3}{n^3\pi^3} & n \text{ odd} \\ \frac{-L^3}{n\pi} & n \text{ even} \end{cases}$$

So the amplitude of the nth mode is

$$B_n = \begin{cases} 32/n^3\pi^3 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

with characteristic frequencies given, as usual, by $\omega_n = \frac{n\pi}{L}\sqrt{\frac{\tau}{\mu}}$.

3.
Solve for the motion $y(x,t)$ of a continuous string (tension τ, mass per unit length μ) with fixed endpoints $y = 0$ at $x = 0$ and $x = L$, when the initial conditions are

$$\begin{aligned} y(x,0) &= A\sin\frac{\pi x}{L} \\ \dot{y}(x,0) &= V\sin\frac{5\pi x}{L}\ , \end{aligned}$$

where A and V are constants.
Solution:
Using the same expansion for the solution as in **1.** and **2.**, we apply the initial conditions:

$$\begin{aligned} y(x,0) &= \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \\ &= A\sin\frac{\pi x}{L}\ , \end{aligned}$$

from which we can see that $B_1 = A$ and all the other B_n are zero. We also have:

$$\begin{aligned} \dot{y}(x,0) &= \sum_{n=1}^{\infty} \omega_n A_n \sin\frac{n\pi x}{L} \\ &= V\sin\frac{5\pi x}{L} \end{aligned}$$

By inspection, $A_5 = \frac{V}{\omega_5}$, and all the other A_n are zero. Thus the solution is

$$y(x,t) = \frac{V}{\omega_5}\sin\omega_5 t\sin\frac{5\pi x}{L} + A\cos\omega_1 t\sin\frac{\pi x}{L}\ ,$$

where as usual $\omega_n = \frac{n\pi}{L}\sqrt{\frac{\tau}{\mu}}$.

4.
A continuous string (tension τ, mass per unit length μ) is attached to fixed supports *infinitely far away*. At $t = 0$ the string satisfies initial conditions

$$\begin{aligned} y(x,0) &= 0 \\ \frac{\partial y}{\partial t}(x,0) &= \alpha\,\delta(x)\ , \end{aligned}$$

where $\delta(x)$ is a Dirac delta function and α is a constant that can be made arbitrarily infinitesimal, so that the string's slope remains small enough for the usual wave equation to apply. This initial condition is appropriate to the string having been struck at $(x = 0, t = 0)$ with a sharp object.

Compute $y(x,t)$ for $t > 0$.
Solution:
From Notes eqn. 14.5, we have:

$$\begin{aligned} y(x,t) &= \frac{1}{2}\big(y_0(x-ct) + y_0(x+ct)\big) \\ &\quad + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(u)\,du \\ &= \frac{1}{2}(0+0) + \frac{1}{2c}\int_{x-ct}^{x+ct} \alpha\,\delta(u)\,du \end{aligned}$$

The integral is $\frac{\alpha}{2c}$ if the interval $(x - ct, x + ct)$ contains 0, zero otherwise. Therefore:

$$y(x,t) = \begin{cases} \frac{\alpha}{2c} & \text{if } -ct < x < ct \\ 0 & \text{otherwise} \end{cases}$$

(Here $c \equiv \sqrt{\frac{\tau}{\mu}}$.)

5.
Show that if ψ and ψ^* are taken as two *independent* field variables, the Lagrangian density

$$\mathcal{L}' = \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi + V\psi^*\psi + \frac{\hbar}{2i}(\psi^*\dot{\psi} - \psi\dot{\psi}^*)$$

(where $\dot{}$ means $\partial/\partial t$ in this context) leads to the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

and its complex conjugate.
Solution:
The above expression for the Lagrangian can be written as:

$$\mathcal{L}' = \frac{\hbar^2}{2m}(\partial_j\psi^*)(\partial_j\psi) + V\psi^*\psi + \frac{\hbar}{2i}(\psi^*\dot{\psi} - \psi\dot{\psi}^*)$$

where we are using the summation convention, and $\partial_j \equiv \frac{\partial}{\partial x_j}$. Our Euler-Lagrange equation looks like:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) + \frac{d}{dx_k}\left(\frac{\partial \mathcal{L}}{\partial(\partial_k\psi)}\right) = \frac{\partial \mathcal{L}}{\partial \psi}$$

where

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) = \frac{\hbar}{2i}\dot{\psi}^*$$
$$\frac{d}{dx_k}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_k\psi\right)}\right) = \frac{\hbar^2}{2m}\frac{d}{dx_k}\left(\partial_k\psi^*\right) = \frac{\hbar^2}{2m}\nabla^2\psi^*$$
$$\frac{\partial \mathcal{L}}{\partial \psi} = V\psi^* - \frac{\hbar}{2i}\dot{\psi}^*$$

Putting all these into the Euler-Lagrange formula, we get

$$\frac{\hbar}{2i}\dot{\psi}^* + \frac{\hbar^2}{2m}\nabla^2\psi^* = V\psi^* - \frac{\hbar}{2i}\dot{\psi}^*$$

Rearrange that to get

$$-\frac{\hbar^2}{2m}\nabla^2\psi^* + V\psi^* = -i\hbar\dot{\psi}^*$$

which is the complex conjugate of the usual Schrödinger equation. If you apply the Euler-Lagrange equation for ψ^*, you'll get the unconjugated Schrödinger equation.

6.
Consider a membrane stretched between *fixed* supports at $x = 0$, $x = L$, $y = 0$, and $y = L$. *Per unit area*, its kinetic and potential energies are

$$T' = \tfrac{1}{2}\sigma\big(\frac{\partial z}{\partial t}\big)^2$$
$$U' = \tfrac{1}{2}\beta\Big(\big(\frac{\partial z}{\partial x}\big)^2 + \big(\frac{\partial z}{\partial y}\big)^2\Big) ,$$

where σ is the membrane's mass per unit area, β is a constant that is inversely proportional to its elasticity, and z is its (normal) displacement.

Apply the Euler-Lagrange equations to obtain a partial differential equation for $z(x, y, t)$. Using a trial solution

$$z(x, y, t) = X(x)\,Y(y)\,T(t) ,$$

find the angular frequencies of vibration for the *five* lowest-frequency normal modes of oscillation.
Solution:
Our Lagrangian is

$$\mathcal{L}' = \frac{1}{2}\sigma\left(\frac{\partial z}{\partial t}\right)^2 - \frac{1}{2}\beta\left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right)$$

Apply the Euler-Lagrange equation:

$$\frac{d}{dx}\frac{\partial \mathcal{L}'}{\partial \frac{\partial z}{\partial x}} + \frac{d}{dy}\frac{\partial \mathcal{L}'}{\partial \frac{\partial z}{\partial y}} + \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \frac{\partial z}{\partial t}} = \frac{\partial \mathcal{L}'}{\partial z}$$
$$-\frac{d}{dx}\left(\beta\frac{\partial z}{\partial x}\right) - \frac{d}{dy}\left(\beta\frac{\partial z}{\partial y}\right) + \frac{d}{dt}\left(\sigma\frac{\partial z}{\partial t}\right) = 0$$
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = 0 ,$$

where $c \equiv \sqrt{\frac{\beta}{\sigma}}$. Substituting in a solution of the form $z(z, y, t) = X(x)Y(y)T(t)$ and dividing through by z yields:

$$\frac{X^{''}}{X} + \frac{Y^{''}}{Y} - \frac{1}{c^2}\frac{T^{''}}{T} = 0$$

By the usual separation of variables reasoning, each term must be separately equal to a constant. Therefore we try a solution of the form

$$X(x) \propto \sin\frac{n\pi x}{L}$$
$$Y(y) \propto \sin\frac{m\pi x}{L}$$
$$T(t) \propto e^{i\omega t}$$

where n and m are positive integers. Note that the coefficients of x are chosen to satisfy the boundary conditions $X(0) = X(L) = Y(0) =$

$Y(L) = 0$. In order to still satisfy the separated D.E., we must have

$$-\left(\frac{n\pi}{L}\right)^2 - \left(\frac{m\pi}{L}\right)^2 + \frac{\omega^2}{c^2} = 0$$
$$\omega^2 = \frac{c^2\pi^2}{L^2}(n^2+m^2)$$

and so the frequencies of the five lowest frequency modes are given by:

$$\omega^2 = \frac{c^2\pi^2}{L^2}\begin{cases} 1^2+1^2=2 \\ 1^2+2^2=5 \\ 2^2+1^2=5 \\ 2^2+2^2=8 \\ 3^2+1^2=10 \\ 1^2+3^2=10 \\ 2^2+3^2=13 \\ 3^2+2^2=13 \end{cases}$$

7. and **8.** (double problem)
The Lagrangian density (per unit volume) for a charge density $\rho(\mathbf{r},t)$ and current density $\mathbf{j}(\mathbf{r},t)$ in the presence of an electromagnetic field $\mathbf{E}(\mathbf{r},t)$, $\mathbf{B}(\mathbf{r},t)$ is

$$\mathcal{L}' = \frac{E^2-B^2}{8\pi} - \rho\phi + \frac{1}{c}\mathbf{j}\cdot\mathbf{A}\ .$$

The first term is the Lagrangian density corresponding to the self-energy of the free field, and the latter terms represent the interaction between fields and charges. The self-energy of the individual (point) charges is infinity in classical theory and is omitted. In the above, $\mathbf{A}$ is the *vector potential* defined by

$$\mathbf{B} = \nabla\times\mathbf{A}$$
$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$$

(Gaussian units are used throughout this problem). If you are familiar with relativistic transformations of electromagnetic fields, you may notice that the above Lagrangian density is *Lorentz invariant*, although not manifestly so.

The homogeneous (charge and current independent) Maxwell equations follow directly from the equations relating $\mathbf{E}$ and $\mathbf{B}$ to the potentials. To complete the picture, using ϕ and the three components of $\mathbf{A}$ as four generalized (field) coordinates, apply the Euler-Lagrange equations to $\mathcal{L}'$ to obtain the two inhomogeneous Maxwell equations

$$\nabla\cdot\mathbf{E} = 4\pi\rho$$
$$\nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j}\ .$$

Solution:
Remember the repeated-index summation convention; we'll be using it a lot. The ith component of the electric field is $E_i = -(\partial_i\phi + \dot{A}_i/c)$. So E^2 is

$$E^2 = E_iE_i = (\partial_i\phi + \dot{A}_i)(\partial_i\phi + \dot{A}_i)\ ,$$

and B^2 is

$$B^2 = (\nabla\times\vec{A})^2 = (\nabla\times\vec{A})_i(\nabla\times\vec{A})_i$$
$$= \epsilon_{ijk}(\partial_jA_k)\epsilon_{ilm}(\partial_lA_m)\ .$$

So the Lagrangian density is

$$\mathcal{L} = \frac{1}{8\pi}\left(\partial_i\phi + \frac{1}{c}\dot{A}_i\right)\left(\partial_i\phi + \frac{1}{c}\dot{A}_i\right) - \frac{1}{8\pi}\epsilon_{ijk}\epsilon_{ilm}(\partial_jA_k)(\partial_lA_m) - \rho\phi + \frac{1}{c}j_iA_i\ .$$

The Euler-Lagrange equation for a coordinate η is

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\eta}}\right) + \frac{d}{dx_a}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_a\eta\right)}\right) - \frac{\partial\mathcal{L}}{\partial\eta} = 0$$

(Note that there is an implied sum over a in the second term.) We have four coordinates: ϕ and the three components of $\vec{A}$. Let's start by setting $\eta = \phi$. Then the first term is zero, and the third term is $\partial\mathcal{L}/\partial\phi = -\rho$. To figure out the second term, note that $\partial_a\phi$ occurs in the Lagrangian density only in the first term, and only when $i = a$. So

$$\frac{\partial\mathcal{L}}{\partial\left(\partial_a\phi\right)} = \frac{1}{4\pi}\left(\partial_a\phi + \frac{1}{c}\dot{A}_a\right)$$

Putting all of this into the Euler-Lagrange equation, we get

$$\frac{1}{4\pi}\frac{d}{dx_a}\left(\partial_a\phi + \frac{1}{c}\dot{A}_a\right) + \rho = 0$$

The first term is just $-(1/4\pi)dE_a/dx_a$, which is $-\nabla\cdot\vec{E}/4\pi$, so this equation is

$$\nabla\cdot\vec{E} = 4\pi\rho$$

Now let's choose as our coordinate an arbitrary component of the vector potential: $\eta = A_b$. Then

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{A}_b}\right) &= \frac{1}{4\pi c}\frac{d}{dt}\left(\partial_b\phi + \frac{1}{c}\dot{A}_b\right)\\ &= -\frac{1}{4\pi c}\frac{dE_b}{dt}\end{aligned}$$

and

$$\begin{aligned}&\frac{d}{dx_a}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_a A_b\right)}\right)\\ &= -\frac{1}{8\pi}\frac{d}{dx_a}\left(\epsilon_{iab}\epsilon_{ilm}\partial_l A_m + \epsilon_{ijk}\epsilon_{iab}\partial_j A_k\right)\\ &= -\frac{1}{4\pi}\epsilon_{iab}\epsilon_{ilm}\partial_a\partial_l A_m\end{aligned}$$

and

$$\frac{\partial\mathcal{L}}{\partial A_b} = \frac{j_b}{c}$$

The second expression above requires some explanation. The term in the Lagrangian density that involves spatial derivatives of $\vec{A}$ has no fewer than five implied summations: i, j, k, l, m are all summed over. The derivative with respect to $\partial_a A_b$ gets nonzero contributions when $(j,k) = (a,b)$ and when $(l,m) = (a,b)$. Those are the two terms in the second line of the second expression above. Those two terms are equal, as you can see by relabeling the remaining dummy indices. (Specifically, relabel j,k to be l,m in the second term.)

So the Euler-Lagrange equation is

$$-\frac{1}{4\pi c}\frac{dE_b}{dt} - \frac{1}{4\pi}\epsilon_{iab}\epsilon_{ilm}\partial_a\partial_l A_m - \frac{j_b}{c} = 0$$

But

$$\begin{aligned}\epsilon_{iab}\epsilon_{ilm}\partial_a\partial_l A_m &= \epsilon_{iab}\partial_a\left(\nabla\times\vec{A}\right)_i\\ &= -\epsilon_{bai}\partial_a B_i\\ &= -\left(\nabla\times\vec{B}\right)_b\end{aligned}$$

So the Euler-Lagrange equation for A_b becomes

$$\frac{1}{c}\frac{dE_b}{dt} - \left(\nabla\times\vec{B}\right)_b + \frac{j_b}{4\pi c} = 0$$

which is just the b component of the second Maxwell equation.

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

ASSIGNMENT 12

Reading:
105 Notes 14.6
Hand & Finch 10.1-10.2

1.
Consider a uniform cube of side L. Inside the cube is a scalar field ϕ that satisfies the wave equation with characteristic wavespeed c. At the surfaces of the cube, ϕ is required to vanish.
(**a**)
Show that for this system the total number of modes of vibration corresponding to frequencies between ν and $\nu + d\nu$ is $4\pi L^3\nu^2 d\nu/c^3$, if $\pi c/L \ll d\nu \ll \nu$.
(**b**)
What would the result be for a (two-dimensional) square?
(**c**)
A (one-dimensional) rod?

2. and **3.** (double credit problem)
Consider a homogeneous isotropic *solid* medium, *i.e.* a medium that, unlike a liquid, is able to resist being twisted (it "supports a shear stress"). The Lagrangian density for such a medium is

$$\mathcal{L}' = \frac{1}{2}\rho\frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t} - \frac{1}{2}\frac{\partial u_i}{\partial x_j}C_{ijkl}\frac{\partial u_k}{\partial x_l}\ ,$$

where summation over repeated indices is (definitely!) implied. In this expression, the field variables are $u_1(x_1, x_2, x_3, t)$, $u_2(x_1, x_2, x_3, t)$, and $u_3(x_1, x_2, x_3, t)$. These describe the (vector) displacement $\mathbf{u}$ of a small element of the solid from its equilibrium position $\mathbf{x}$. (The *strain* is obtained by taking spatial derivatives of $\mathbf{u}$.) The mass density of the solid is ρ, which for small values of $\mathbf{u}$ can be approximated as a constant. C_{ijkl} is the "fourth-rank tensor of elasticity".

Exploiting the homogeneous medium's isotropy, one can show that the most general form for C_{ijkl} is

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\ ,$$

where λ and μ, the so-called "Lamé constants", determine all 81 of its elements. The inverse of the *compression modulus* λ is proportional to the compressibility of the medium, and the inverse of the *shear modulus* μ is proportional to the extent to which the medium can be twisted.

Notice that the Lagrangian density for a solid medium could in principle depend on 19 variables (3 field variables, 3×4 derivatives of 3 field variables with respect to 4 independent variables, and 4 independent variables). In practice, our Lagrangian density has no dependence on the first and last category, so it is a function of only 12 variables.

Use the Euler-Lagrange equations for this Lagrangian density to derive the wave equations for compression waves ($\nabla \times \mathbf{u} = 0$) and for shear waves ($\nabla \cdot \mathbf{u} = 0$) in the solid. Obtain the phase velocity c for both cases, in terms of λ, μ, and ρ. Notice that an earthquake can propagate with more than one velocity!

4.
Consider an infinitely long continuous string in which the tension is τ. A mass M is attached to the string at $x = 0$. If a sinusoidal wave train with velocity ω/k is incident from the left, analyze the reflection and transmission that occur at $x = 0$. Define the reflection coefficient $R \equiv |\mathcal{R}|^2$ and the transmission coefficient $T \equiv |\mathcal{T}|^2$, where $\mathcal{R}$ and $\mathcal{T}$ are the reflected and transmitted amplitude ratios discussed in Lecture Notes section 14.6.

Show that R and T are given by $R = \sin^2\theta$ and $T = \cos^2\theta$, where $\tan\theta = M\omega^2/2k\tau$. [*Hint*: Consider carefully the boundary condition on the derivatives of the wave functions at $x = 0$.]

5.
Hand & Finch, Problem 10.1 (stability in a central force)

6.
Hand & Finch, p. 397, *Question* 6 (upside-down pendulum)

7. and **8.** (double credit problem)
Hand & Finch, Problem 10.9 (a)-(d) *only* (how does a child pump a swing?)

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO PROBLEM SET 12

Solutions by T. Bunn and J. Barber

Reading:
105 Notes 14.6
Hand & Finch 10.1-10.2

1.
Consider a uniform cube of side L. Inside the cube is a scalar field ϕ that satisfies the wave equation with characteristic wavespeed c. At the surfaces of the cube, ϕ is required to vanish.

(a)
Show that for this system the total number of modes of vibration corresponding to frequencies between ν and $\nu + d\nu$ is $4\pi^2 L^3 \nu^2 d\nu/c^3$, if $\pi c/L \ll d\nu \ll \nu$.

Solution:
Let's first figure out what the normal modes look like. The wave equation inside the box is

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0$$

You can solve this equation by separation of variables: Assume that the solution is of the form $\phi(x, y, z, t) = X(x)Y(y)Z(z)T(t)$, and you'll find that the solutions are of the form

$$\phi \propto e^{i(k_x x + k_y y + k_z z - \omega t)} \text{ , where}$$
$$\omega^2 \equiv c^2(k_x^2 + k_y^2 + k_z^2) \ .$$

In order to satisfy the boundary condition $\phi = 0$ on the edges of the box, the complex exponentials in x, y, z must all be sines, not cosines, and the numbers k_x, k_y, k_z must all be integer multiples of π/L. So the normal modes are

$$\phi \propto e^{i\omega t} \sin\frac{l\pi x}{L} \sin\frac{m\pi y}{L} \sin\frac{n\pi z}{L}$$

with l, m, n positive integers. The frequency of a given mode is $\nu = \omega/2\pi = (c/2\pi)|\vec{k}|$, where $\vec{k} = (k_x, k_y, k_z) = \left(\frac{l\pi}{L}, \frac{m\pi}{L}, \frac{n\pi}{L}\right)$. We need to find the number of modes between ν and $\nu + d\nu$, but since ν and $|\vec{k}|$ are proportional, let's find the number between $|\vec{k}|$ and $|\vec{k}| + d|\vec{k}|$ instead.

Let $N(k)$ be the number of modes whose wave vector $\vec{k}$ is of length less than k. If you picture the wave vectors as points in three-dimensional space, $N(k)$ is the number of points inside of one octant of a sphere of radius k. (It's only one octant because negative values of the integers l, m, n don't lead to physically distinct states.) Using the usual formula for the volume of the sphere, we get

$$N(k) = \frac{1}{8} \cdot \frac{4\pi}{3} k^3 \cdot \begin{pmatrix} \text{density of wave vectors} \\ \text{in } k\text{-space} \end{pmatrix}$$

The last term is simply the number of allowed $\vec{k}$'s per unit volume in k-space. The allowed k-vectors are spaced on a cubic lattice of side π/L, so there is one vector per volume $(\pi/L)^3$. The density is therefore $(L/\pi)^3$ vectors per unit volume, and

$$N(k) = \frac{L^3 k^3}{6\pi^2}$$

Use the formula $k = 2\pi\nu/c$ to get $N(\nu) = 4\pi L^3 \nu^3/3c^3$ for the number of modes with frequency less than ν. The number of modes between ν and $\nu + d\nu$ is the differential of this:

$$dN = \frac{4\pi L^3}{c^3} \nu^2 \, d\nu$$

(b)
What would the result be for a (two-dimensional) square?

Solution:
In two dimensions, use the formula for the area of a one quadrant of a circle instead of the volume of one octant of a sphere, and use $(L/\pi)^2$ instead of $(L/\pi)^3$ for the density in k-space. Then

$$N(k) = \frac{1}{4}\pi k^2 \left(\frac{L}{\pi}\right)^2$$

Convert to frequency and differentiate to get

$$dN = \frac{2\pi L^2}{c^2}\,\nu\,d\nu$$

(**c**)
A (one-dimensional) rod?
Solution:
In one dimension, k-space is just a line, and so instead of the volume or the area, you just have the length. The same process gives

$$dN = \frac{2L}{c}\,d\nu$$

2. and **3.** (double credit problem)
Consider a homogeneous isotropic *solid* medium, *i.e.* a medium that, unlike a liquid, is able to resist being twisted (it "supports a shear stress"). The Lagrangian density for such a medium is

$$\mathcal{L}' = \frac{1}{2}\rho\frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t} - \frac{1}{2}\frac{\partial u_i}{\partial x_j}C_{ijkl}\frac{\partial u_k}{\partial x_l}\ ,$$

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Use the Euler-Lagrange equations for this Lagrangian density to derive the wave equations for compression waves ($\nabla\times\mathbf{u}=0$) and for shear waves ($\nabla\cdot\mathbf{u}=0$) in the solid. Obtain the phase velocity c for both cases, in terms of λ, μ, and ρ. Notice that an earthquake can propagate with more than one velocity!
Solution:
Start from the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}'}{\partial\dot{u}_n}\right) + \frac{d}{dx_m}\left(\frac{\partial\mathcal{L}'}{\partial\left(\frac{\partial u_n}{\partial x_m}\right)}\right) - \frac{\partial\mathcal{L}'}{\partial u_n} = 0$$

The first term just gives $\rho\ddot{u}_n$, and the third term is zero. Let's figure out the second term.

$$\begin{aligned}\frac{\partial\mathcal{L}'}{\partial\left(\frac{\partial u_n}{\partial x_m}\right)} &= -\frac{1}{2}C_{nmkl}\frac{\partial u_k}{\partial x_l} - \frac{1}{2}C_{ijnm}\frac{\partial u_i}{\partial x_j}\\ &= -\frac{1}{2}\left(C_{nmij}+C_{ijnm}\right)\frac{\partial u_i}{\partial x_j}\\ &= -\lambda\delta_{nm}\frac{\partial u_i}{\partial x_i} - \mu\left(\frac{\partial u_n}{\partial x_m}+\frac{\partial u_m}{\partial x_n}\right)\ .\end{aligned}$$

Taking the derivative with respect to x_m, we get

$$\begin{aligned}&\frac{d}{dx_m}\left(\frac{\partial\mathcal{L}'}{\partial\left(\frac{\partial u_n}{\partial x_m}\right)}\right) =\\ &= -\lambda\frac{\partial^2 u_i}{\partial x_n\,\partial x_i} - \mu\frac{\partial^2 u_n}{\partial x_m\,\partial x_m} - \mu\frac{\partial^2 u_m}{\partial x_m\,\partial x_n}\\ &= -(\lambda+\mu)\left(\nabla\left(\nabla\cdot\vec{u}\right)\right)_n - \mu\nabla^2 u_n\ .\end{aligned}$$

So the full Euler-Lagrange equation is

$$\rho\frac{\partial^2\vec{u}}{\partial t^2} - (\lambda+\mu)\nabla\left(\nabla\cdot\vec{u}\right) - \mu\nabla^2\vec{u} = 0$$

Now let's use this to get the wave equation for compression and shear waves. First, suppose $\nabla\times\vec{u}=0$, so we have a compression wave. Then we can perform a trick to get rid of the unwanted $\nabla(\nabla \cdot \vec{u})$ term in the wave equation: If $\nabla \times \vec{u} = 0$, then $\nabla \times (\nabla \times \vec{u}) = 0$. But there's a vector identity that says

$$\nabla \times (\nabla \times \vec{u}) = \nabla(\nabla \cdot \vec{u}) - \nabla^2 \vec{u}$$

so $\nabla(\nabla \cdot \vec{u}) = \nabla^2 \vec{u}$. Then the wave equation becomes

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} - (\lambda + 2\mu)\nabla^2 \vec{u} = 0$$

That's the wave equation for compression waves. The wave speed is given by $c_c^2 = (\lambda + 2\mu)/\rho$.

Shear waves are easier. Since $\nabla \cdot \vec{u} = 0$, the Euler-Lagrange equation becomes

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} - \mu \nabla^2 \vec{u} = 0$$

and the wave speed is given by $c_s^2 = \mu/\rho$.

4.
Consider an infinitely long continuous string in which the tension is τ. A mass M is attached to the string at $x = 0$. If a sinusoidal wave train with velocity ω/k is incident from the left, analyze the reflection and transmission that occur at $x = 0$. Define the reflection coefficient $R \equiv |\mathcal{R}|^2$ and the transmission coefficient $T \equiv |\mathcal{T}|^2$, where $\mathcal{R}$ and $\mathcal{T}$ are the reflected and transmitted amplitude ratios discussed in Lecture Notes section 14.6.

Show that R and T are given by $R = \sin^2\theta$ and $T = \cos^2\theta$, where $\tan\theta = M\omega^2/2k\tau$. [*Hint*: Consider carefully the boundary condition on the derivatives of the wave functions at $x = 0$.]
Solution:
Call the displacement y_1 for $x < 0$ and y_2 for $x > 0$. Then y_1 and y_2 are of the form

$$\begin{aligned} y_1(x,t) &= \mathrm{Re}\big(Ae^{ikx-i\omega t} + Be^{-ikx-i\omega t}\big) \\ y_2(x,t) &= \mathrm{Re}\big(Ce^{ikx-i\omega t}\big)\ , \end{aligned}$$

where A, B, C are the (complex) amplitudes of the incident, reflected, and transmitted waves, respectively. The requirement that $y_1(0,t) = y_2(0,t)$ yields a real equation; we choose to solve the complex equation of which that equation is the real part. It is

$$A + B = C\ .$$

If the point mass weren't there, we would impose the condition that the forces just to the left and right of $x = 0$ add up to zero. That requirement is not satisfied here, since the point mass at $x = 0$ is accelerating, and so the net force on it must be nonzero. Instead, we can apply Newton's second law: $F_{1y} + F_{2y} = M\ddot{y}$. Here F_{1y} means the y-component of the force exerted by the left half of the string on the mass. Clearly this is $-\tau\sin\phi$, where ϕ is the angle the left half of the string makes with the horizontal. The slope of the string is $dy_1/dx = \tan\phi$, but for small angles $\sin\phi \approx \tan\phi$, so we can say $F_{1y} = -\tau\, dy_1/dx$. Similarly, $F_{2y} = \tau\, dy_2/dx$. So our second boundary condition is

$$\tau\left(\frac{dy_2}{dx} - \frac{dy_1}{dx}\right) = M\ddot{y}$$

(The y on the right-hand-side can be either y_1 or y_2, since they're equal at $x = 0$.) Using our expressions for y_1 and y_2, we get

$$ik\tau(C - A + B) = -M\omega^2 C$$

Solving these two equations, we get

$$\begin{aligned} B &= \frac{-M\omega^2 A}{M\omega^2 + 2ik\tau} \\ C &= \frac{2ik\tau A}{M\omega^2 + 2ik\tau} \end{aligned}$$

The reflection and transmission coefficients are

$$\begin{aligned} R &= \left|\frac{B}{A}\right|^2 = \frac{M^2\omega^4}{M^2\omega^4 + 4k^2\tau^2} \\ T &= \left|\frac{C}{A}\right|^2 = \frac{4k^2\tau^2}{M^2\omega^4 + 4k^2\tau^2} \end{aligned}$$

If we define $T = \cos^2\theta$ and $R = \sin^2\theta$, then $\tan\theta = \sqrt{\frac{R}{T}} = \frac{M\omega^2}{2k\tau}$.

5.
Hand & Finch, Problem 10.1 (stability in a central force)

Solution:
(**a**)
From Hand and Finch equation 4.41, we have

$$\mu\ddot{r} = \frac{l^2}{\mu r^3} + \frac{n\beta}{r^{n+1}}$$

To find the radius r_o of a stationary circular orbit, we set $\ddot{r} = 0$.

$$0 = \frac{l^2}{\mu r_o^3} + \frac{n\beta}{r_o^{n+1}}$$
$$r_o = \left(-\frac{n\beta\mu}{l^2}\right)^{\frac{1}{n-2}}$$

In order for this quantity to exist, we must have $n\beta < 0$ and $n \neq 2$. (If $n = 2$, we can have (not stable) circular orbits at any r_0 only if $\beta = -l^2/2\mu$.)
(**b**)
Let $r = r_o + \delta r(t)$, where $\delta r \ll r_o$. Plugging this into the above equation yields

$$\mu\ddot{\delta r} = \frac{l^2}{\mu(r_o + \delta r)^3} + \frac{n\beta}{(r_o + \delta r)^{n+1}} \; .$$

Now Taylor expand around $\delta r = 0$ to get

$$\mu\ddot{\delta r} = -\left(\frac{3l^2}{\mu r_o^4} + \frac{(n+1)n\beta}{r_o^{n+2}}\right)\delta r$$
$$\ddot{\delta r} = -\frac{l^2}{\mu}\left(-\frac{n\beta\mu}{l^2}\right)^{\frac{4}{2-n}}(2-n)\delta r$$

In order for the circular orbit to be stable, we must have the coefficient of δr in the above expression be < 0. Since we already know that $n\beta < 0$, this requires that $n < 2$.

6.
Hand & Finch, p. 397, *Question* 6 (upside-down pendulum)
Solution:
With our origin at the point of support of the pendulum, we have $x = l\sin\theta$ and $y = Y(t) - l\cos\theta$. Since we are interested in the case where θ is near π, we make the substitution $\theta = \pi + \psi$. Differentiating with respect to time yields

$$\dot{x} = -l\dot{\psi}\cos\psi$$
$$\dot{y} = \dot{Y} - l\dot{\psi}\sin\psi \; .$$

And so the kinetic and potential energies are:

$$\begin{aligned} T &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2) \\ &= \frac{m}{2}\left(l^2\dot{\psi}^2 - 2l\dot{Y}\dot{\psi}\sin\psi + \dot{Y}^2\right) \\ U &= mgy = m\omega_o^2 l\,(Y + l\cos\psi) \end{aligned}$$

where $\omega_o = \sqrt{\frac{g}{l}}$. Applying the Euler-Lagrange equation yields

$$l\ddot{\psi} - \left(\ddot{Y} + \omega_o^2 l\right)\sin\psi = 0 \; .$$

Now, following Hand & Finch, we plug in $Y(t) = Y_o\cos\Omega t$, and define $a \equiv \left(\frac{2\omega_o}{\Omega}\right)^2$ and $q \equiv \frac{2Y_o}{l}$, to give

$$\frac{4}{\Omega^2}\ddot{\psi} - (a - 2q\cos\Omega t)\sin\psi = 0 \; .$$

Finally, if we define $\tau = \frac{\Omega t}{2}$, then this can be slightly simplified to obtain:

$$\frac{d^2\psi}{d\tau^2} - (a - 2q\cos\Omega t)\sin\psi = 0$$

which differs from Hand & Finch equation 10.36 only by a single $-$ sign. Note that *no* approximations were made in this derivation.

7. and **8.** (double credit problem)
Hand & Finch, Problem 10.9 (a)-(d) *only* (how does a child pump a swing?)
Solution:
(**a**)
The Lagrangian for this system is

$$\mathcal{L} = \frac{m}{2}\left(\dot{l}^2 + l^2\dot{\theta}^2\right) + mgl\cos\theta \; .$$

Apply the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0$$
$$\frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta = 0$$

If we let $\theta \ll 1$, then this becomes

$$\frac{d}{dt}(l^2\dot{\theta}) + gl\theta = 0 \ .$$

(b)
If we now substitute in $l(t) = \bar{l}(1+\delta(t))$ and $\tau = \sqrt{\frac{g}{l}}t$, the above expression becomes:

$$\frac{d}{d\tau}\left((1+\delta)^2\dot{\theta}\right) + (1+\delta)\theta = 0$$

Making the following substitutions

$$\begin{aligned}\theta &= \frac{\phi}{l(t)} = \frac{1}{\bar{l}}\frac{\phi}{(1+\delta)} \\ \dot{\theta} &= \frac{1}{\bar{l}}\left(\frac{\dot{\phi}}{1+\delta} - \frac{\phi\dot{\delta}}{(1+\delta)^2}\right)\end{aligned}$$

yields

$$\begin{aligned}\frac{1}{\bar{l}}\frac{d}{d\tau}\left((1+\delta)^2\left(\frac{\dot{\phi}}{1+\delta} - \frac{\phi\dot{\delta}}{(1+\delta)^2}\right)\right)& \\ + (1+\delta)\frac{\phi}{\bar{l}(1+\delta)} &= 0 \\ \frac{d}{d\tau}\left((1+\delta)\dot{\phi} - \phi\dot{\delta}\right) + \phi &= 0 \\ \ddot{\phi} + \left(\frac{1-\ddot{\delta}}{1+\delta}\right)\phi &= 0\end{aligned}$$

(c)
For small δ,

$$\left(\frac{1-\ddot{\delta}}{1+\delta}\right) \approx (1-\ddot{\delta})(1-\delta) \approx 1 - \delta - \ddot{\delta} \ ,$$

in which case the DE becomes

$$\ddot{\phi} + \phi = (\delta + \ddot{\delta})\phi \ .$$

Now, let

$$\begin{aligned}\delta &= \delta_0 \cos 2\tau \\ \ddot{\delta} &= -4\delta_0 \cos 2\tau \\ \phi &= Al(0)\cos\tau + \mathcal{O}(\delta_0)\end{aligned}$$

We obtain as an equation for ϕ

$$\begin{aligned}\ddot{\phi} + \phi &= (\delta_0 \cos 2\tau - 4\delta_0 \cos 2\tau)\ Al(0)\cos\tau \\ &= -3\delta_0 Al(0)\cos 2\tau\cos\tau\end{aligned}$$

Using the trig identity $\cos\tau\ \cos 2\tau = \frac{1}{2}(\cos\tau + \cos 3\tau)$ gives (to first order in δ_0) the result:

$$\ddot{\phi} + \phi = -\frac{3}{2}\delta_0 Al(0)\left(\cos\tau + \cos 3\tau\right)$$

(d)
The homogenous solution of the above DE is of the form

$$\phi_h = B\sin\tau + C\cos\tau \ ,$$

where B and C will need to be chosen to satisfy the initial conditions (we will wait to do this until we have the particular solution as well). The particular solution to the above DE is the sum of the particular solutions ϕ_1 and ϕ_3 to the differential equations

$$\begin{aligned}\ddot{\phi}_1 + \phi_1 &= -\frac{3}{2}\delta_0 Al(0)\cos\tau \\ \ddot{\phi}_3 + \phi_3 &= -\frac{3}{2}\delta_0 Al(0)\cos 3\tau \ .\end{aligned}$$

The equation for ϕ_3 is easier to solve, so let's do it first. Substitute $\phi_3 \equiv D\cos 3\tau$ to obtain

$$\begin{aligned}(-9D + D)\cos 3\tau &= -\frac{3}{2}\delta_0 Al(0)\cos 3\tau \\ D &= \frac{3}{16}\delta_0 Al(0) \ .\end{aligned}$$

Turning to the equation for ϕ_1, if we were to substitute $\phi_1 = E\cos\tau$ in analogy to the method we used for ϕ_1, the LHS would vanish and the equation would not be satisfied. Instead (more or less by trial and error), we substitute $\phi_1 = F\tau\sin\tau$. (If there is a rationale, it is that the extra factor of τ can be expected to destroy the cancellation on the LHS, allowing some harmonic function of τ to survive.)

$$\begin{aligned}\left(\frac{d^2}{d\tau^2} + 1\right)(F\tau\sin\tau) &= -\frac{3}{2}\delta_0 Al(0)\cos\tau \\ \frac{d}{d\tau}(\sin\tau + \tau\cos\tau) + \tau\sin\tau &= -\frac{3\delta_0 Al(0)}{2F}\cos\tau \\ 2\cos\tau &= -\frac{3\delta_0 Al(0)}{2F}\cos\tau \\ F &= -\frac{3}{4}\delta_0 Al(0) \ .\end{aligned}$$

Putting it all together, the general solution for ϕ is

$$\begin{aligned}\phi &= \phi_h + \phi_1 + \phi_3 \\ &= B\sin\tau + C\cos\tau \\ &+ \delta_0 A l(0)\big(\tfrac{3}{16}\cos 3\tau - \tfrac{3}{4}\tau\sin\tau\big) \ .\end{aligned}$$

The variable ϕ is the product of two time-dependent functions:

$$\begin{aligned}\phi(\tau) &= \theta(\tau) l(\tau) \\ &= \theta(\tau)(\bar{l} + \delta_0\cos 2\tau) \ .\end{aligned}$$

Taking its derivative with respect to τ,

$$\dot{\phi}(\tau) = \dot{\theta}(\tau)(\bar{l} + \delta_0\cos 2\tau) - 2\theta(\tau)\delta_0\sin 2\tau \ .$$

Applying the initial conditions $\theta(0) = A$, $\dot{\theta}(0) = 0$, we obtain the initial conditions on ϕ:

$$\begin{aligned}\phi(0) &= A(\bar{l} + \delta_0) \\ &= A l(0) \\ \dot{\phi}(0) &= 0 \ .\end{aligned}$$

Finally we use these initial conditions to determine B and C:

$$\begin{aligned}A l(0) &= \phi(0) \\ &= C + \delta_0 A l(0)\big(\tfrac{3}{16}\big) \\ A l(0)\big(1 - \tfrac{3}{16}\delta_0\big) &= C \\ 0 &= \dot{\phi}(0) \\ &= -B \ .\end{aligned}$$

Plugging these values for B and C into the general solution for ϕ, we obtain the complete expresson for $\phi(\tau)$:

$$\begin{aligned}\phi(\tau) &= A l(0)\left(1 - \tfrac{3}{16}\delta_0\right)\cos\tau \\ &+ \tfrac{3}{16} A l(0)\delta_0\cos 3\tau - \tfrac{1}{4} A l(0)\delta_0\tau\sin\tau \ .\end{aligned}$$

Clearly, the coefficient of the last term is increasing in magnitude linearly with time.

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ASSIGNMENT 13

Reading:
Hand & Finch:
10.4 (Introduction; Harmonic Analysis; Hysteresis)
11 (Introduction and Overview)
11.1
11.11 (Introduction; Breaking the Symmetry; Period Doubling and the Onset of Chaos)
11.13

1.
Hand & Finch, p. 403, *Question* 7 (using symmetry). Note the typo in the statement of this problem: (b) should read $t \to t + \pi/\omega$, not $t \to t + \pi$.

2.
Hand & Finch Problem 10.11 (a) and (b) only (asteroid perturbed by Jupiter)

3.
Using Hand & Finch Eqs. (9.18) and (9.19), derive Eq. (9.20).

4.
Hand & Finch, p. 427, *Question* 1 (Hamiltonian)

5.
Hand & Finch, p. 454, *Question* 12 (phase space flow equations)

6.
Hand & Finch, p. 455, *Question* 13 (symmetry breaking)

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO PROBLEM SET 13
Solutions by J. Barber

Reading:
Hand & Finch:
10.4 (Introduction; Harmonic Analysis; Hysteresis)
11 (Introduction and Overview)
11.1
11.11 (Introduction; Breaking the Symmetry; Period Doubling and the Onset of Chaos)
11.13

1.
Hand & Finch, p. 403, *Question* 7 (using symmetry). Note the typo in the statement of this problem: (b) should read $t \to t + \pi/\omega$, not $t \to t + \pi$.
Solution:
With $Q = \infty$ (no damping), equation 10.60 becomes $\ddot{q} + q + \epsilon q^3 = f \cos \omega t$.
(**a**)
Under the transformation $t \to -t$, $\ddot{q} \to \frac{1}{(-1)^2}\ddot{q} = \ddot{q}$, and $\cos \omega t \to \cos(-\omega t) = \cos \omega t$. This leaves the equation unchanged, and so Eq. 10.60 is invariant under this transformation.
(**b**)
Under the transformation $t \to t + \frac{\pi}{\omega}$, $q \to -q$, nothing happens to the time derivatives, ie $\frac{d}{dt} \to \frac{d}{dt}$. Taking this into account, the transformed equation is

$$\begin{aligned} -\ddot{q} - q + \epsilon(-q)^3 &= f \cos \omega(t + \frac{\pi}{\omega}) \\ -(\ddot{q} + q + \epsilon q^3) &= f \cos(\omega t + \pi) = -f \cos \omega t \\ \ddot{q} + q + \epsilon q^3 &= f \cos \omega t \end{aligned}$$

So 10.60 is invariant under this transformation.
Solutions to 10.60 must be invariant under the same transformations. So, if we guess a solution of the form $q(t) = \sum_{n=1}^{\infty} (A_n \cos n\omega t + B_n \sin n\omega t)$, we must have (by (**a**)):

$$\sum_{n=1}^{\infty} (A_n \cos n\omega t - B_n \sin n\omega t) = q(-t) = q(t) = \sum_{n=1}^{\infty} (A_n \cos n\omega t + B_n \sin n\omega t)$$

which implies that $\sum_{n-1}^{\infty} B_n \sin n\omega t = 0$. This can only be true for all t if all $B_n = 0$.
By (**b**):

$$\begin{aligned} -\sum_{n=1}^{\infty} A_n \cos n\omega(t + \frac{\pi}{\omega}) = -q(t + \frac{\pi}{\omega}) = q(t) &= \sum_{n=1}^{\infty} A_n \cos n\omega t \\ -\sum_{n=1}^{\infty} (-1)^n A_n \cos n\omega t &= \sum_{n=1}^{\infty} A_n \cos n\omega t \\ \sum_{n=1}^{\infty} (1 + (-1)^n) A_n \cos n\omega t &= 0 \end{aligned}$$

The quantity $1+(-1)^n$ is 2 for n even and 0 for n odd. This implies that $2\sum_{n=2,4,6...}^{\infty} A_n \cos n\omega t = 0$, which can only be true for all t if $A_n = 0$ for all even n. Thus $q(t) = \sum_{n=1,3,5...}^{\infty} A_n \cos n\omega t$.

2.
Hand & Finch Problem 10.11 (a) and (b) only (asteroid perturbed by Jupiter)
Solution:
(**a**)
The Lagrangian for this problem, taking into account the perturbation due to Jupiter, is:

$$\mathcal{L} = \frac{m_a}{2}\left(\dot{r}_a^2 + r_a^2\dot{\phi}_a^2\right) + \frac{GM_S m_a}{r_a} + \frac{GM_J m_a r_a}{r_J^2}\cos(\phi_a - \phi_J)$$

Let r_J =constant, and assume the angles ϕ_a and ϕ_J start off in phase, ie $\phi_a = \omega_a t$, $\phi_J = \omega_J t$. Then $\phi_J = \frac{\omega_J}{\omega_a}\phi_a$. If we now write the Lagrangian in terms of $u \equiv \frac{1}{r_a}$, with $\dot{r}_a = -\frac{\dot{u}}{u^2}$, we get

$$\mathcal{L} = \frac{m_a}{2}\left(\frac{\dot{u}^2}{u^4} + \frac{1}{u^2}\dot{\phi}_a^2\right) + GM_S m_a u + \frac{GM_S m_a}{r_J^2}\frac{1}{u}\cos x\phi_a$$

where $x = 1 - \frac{\omega_J}{\omega_a}$. The Euler-Lagrange equation for u is:

$$\frac{\ddot{u}}{u^4} - 2\frac{\dot{u}^2}{u^5} + \frac{1}{u^3}\dot{\phi}_a^2 + GM_S + \frac{GM_J}{r_J^2}\frac{1}{u^2}\cos x\phi_a = 0$$

Next, if we treat the effect of Jupiter as a *small* perturbation, then the asteroid's angular momentum $l = m_a\dot{\phi}_a r_a^2$ is still conserved. We can then change the independent variable from t to ϕ by using the fact that $\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \frac{lu^2}{m_a}\frac{d}{d\phi}$. Then

$$\frac{\ddot{u}}{u^4} = \frac{1}{u^4}\frac{lu^2}{m_a}\frac{d}{d\phi}\left(\frac{lu^2}{m_a}\frac{du}{d\phi}\right) = \frac{l^2}{m_a^2}\left(\frac{d^2u}{d\phi^2} + \frac{1}{u}\left(\frac{du}{d\phi}\right)^2\right)$$

and

$$\frac{\dot{u}^2}{u^5} = \frac{1}{u^5}\left(\frac{lu^2}{m_a}\frac{du}{d\phi}\right)^2 = \frac{l^2}{m_a^2 u}\left(\frac{du}{d\phi}\right)^2$$

So our equation of motion becomes

$$\frac{d^2u}{d\phi^2} + u - \frac{GM_S m_a^2}{l^2} + \frac{M_J}{M_S}\frac{GM_S m_a^2}{l^2}\left(\frac{1}{ur_J}\right)^2\cos x\phi_a = 0\,.$$

If we define $p \equiv a(1-\epsilon^2) = \frac{l^2}{GM_S m_a^2}$, then this can be written as:

$$\frac{d^2u}{d\phi^2} + u - \frac{1}{p} + \frac{M_J}{M_S}\frac{1}{p}\left(\frac{1}{ur_J}\right)^2\cos x\phi_a = 0$$

(**b**)
If we let $u = \frac{1}{p} + \delta u$, with $\delta u << \frac{1}{p}$, then our equation becomes:

$$\frac{d^2\delta u}{d\phi^2} + \delta u + \frac{1}{p}\frac{M_J}{M_S}\left(\frac{1}{\left(\frac{1}{p} + \delta u\right) r_J}\right)^2 \cos x\phi = 0$$

$$\frac{d^2\delta u}{d\phi^2} + \delta u + \frac{p}{r_J^2}\frac{M_J}{M_S}\frac{1}{(1 + p\delta u)^2}\cos x\phi = 0$$

$$\frac{d^2\delta u}{d\phi^2} + \left(1 - \frac{2p^2}{r_J^2}\frac{M_J}{M_S}\cos x\phi\right)\delta u + \frac{p}{r_J}\frac{M_J}{M_S}\cos x\phi = 0$$

where in the last step we have Taylor expanded for small $p\,\delta u$. The last term is a small perturbation that averages to zero over an orbit, so we can neglect it. Furthermore, if we define $\tau \equiv \frac{x\phi}{2}$, then we obtain

$$\frac{d^2\delta u}{d\tau^2} + \left(\frac{4}{x^2} - 2\left(\frac{2p}{x r_J}\right)^2\frac{M_J}{M_S}\cos 2\tau\right)\delta u = 0$$

$$\frac{d^2\delta u}{d\tau^2} + (a - 2q\cos 2\tau)\,\delta u = 0$$

where a and q are as defined in the problem.

3.
Using Hand & Finch Eqs. (9.18) and (9.19), derive Eq. (9.20).
Solution:

$$x_1 = l\sin\phi_1 \quad\to\quad \dot{x}_1 = l\dot{\phi}_1\cos\phi_1 \qquad\qquad y_1 = -l\cos\phi_1 \quad\to\quad \dot{y}_1 = l\dot{\phi}_1\sin\phi_1$$

$$x_2 = x_1 + l\sin(\phi_1+\phi_2) \quad\to\quad \dot{x}_2 = l\dot{\phi}_1\cos\phi_1 + l(\dot{\phi}_1+\dot{\phi}_2)\cos(\phi_1+\phi_2)$$

$$y_2 = y_1 - l\cos(\phi_1+\phi_2) \quad\to\quad \dot{y}_2 = l\dot{\phi}_1\sin\phi_1 + l(\dot{\phi}_1+\dot{\phi}_2)\sin(\phi_1+\phi_2)$$

$$\begin{aligned} V = mg(y_1+y_2) &= mg\left(-l\cos\phi_1 - l\cos\phi_1 - l\cos(\phi_1+\phi_2)\right) \\ &= -mgl\left(2\cos\phi_1 + \cos(\phi_1+\phi_2)\right) \end{aligned}$$

$$\begin{aligned} T &= \frac{m}{2}\Big((l\dot{\phi}_1\cos\phi_1)^2 + (l\dot{\phi}_1\sin\phi_1)^2 + (l\dot{\phi}_1\cos\phi_1 + l(\dot{\phi}_1+\dot{\phi}_2)\cos(\phi_1+\phi_2))^2 \\ &\quad + (l\dot{\phi}_1\sin\phi_1 + l(\dot{\phi}_1+\dot{\phi}_2)\sin(\phi_1+\phi_2))^2\Big) \\ &= \frac{ml^2}{2}\Big(\dot{\phi}_1^2(\cos^2\phi_1+\sin^2\phi_1) + \dot{\phi}_1^2(\cos^2\phi_1+\sin^2\phi_1) \\ &\quad + 2\dot{\phi}_1(\dot{\phi}_1+\dot{\phi}_2)(\cos\phi_1\cos(\phi_1+\phi_2) + \sin\phi_1\sin(\phi_1+\phi_2)) \\ &\quad + (\dot{\phi}_1+\dot{\phi}_2)^2(\cos^2(\phi_1+\phi_2) + \sin^2(\phi_1+\phi_2))\Big) \\ &= \frac{ml^2}{2}\left(2\dot{\phi}_1^2 + 2\dot{\phi}_1(\dot{\phi}_1+\dot{\phi}_2)\cos\phi_2 + (\dot{\phi}_1+\dot{\phi}_2)^2\right) \end{aligned}$$

4.
Hand & Finch, p. 427, *Question* 1 (Hamiltonian)
Solution:
We make the change of notation $\phi_1 \to \alpha$ and $\phi_2 \to \beta$. Let $\mathcal{L}' = \frac{\mathcal{L}}{mgl}$ and let $t \to \sqrt{\frac{g}{l}}\, t$. Then:

$$\mathcal{L}' = \frac{T-V}{mgl} = \frac{1}{2}\left(2\dot{\alpha}^2 + 2\dot{\alpha}(\dot{\alpha}+\dot{\beta})\cos\beta + (\dot{\alpha}+\dot{\beta})^2\right) + 2\cos\alpha + \cos(\alpha+\beta)$$

Find conjugate momenta:

$$\begin{aligned}
l_\alpha &= \frac{\partial \mathcal{L}'}{\partial \dot{\alpha}} = 2\dot{\alpha} + (2\dot{\alpha}+\dot{\beta})\cos\beta + \dot{\alpha} + \dot{\beta} \\
&= (3 + 2\cos\beta)\dot{\alpha} + (1+\cos\beta)\dot{\beta} \\
l_\beta &= \frac{\partial \mathcal{L}'}{\partial \dot{\beta}} = \dot{\alpha}\cos\beta + \dot{\alpha} + \dot{\beta} \\
&= (1+\cos\beta)\dot{\alpha} + \dot{\beta}
\end{aligned}$$

From these one can solve for $\dot{\alpha}$ and $\dot{\beta}$ to yield:

$$\begin{aligned}
\dot{\alpha} &= \frac{l_\alpha - (1+\cos\beta)l_\beta}{3 + 2\cos\beta - (1+\cos\beta)^2} = 2\frac{l_\alpha - (1+\cos\beta)l_\beta}{3 - 2\cos 2\beta} \\
\dot{\beta} &= \frac{(3+2\cos\beta)l_\beta - (1+\cos\beta)l_\alpha}{3 + 2\cos\beta - (1+\cos\beta)^2} = 2\frac{(3+2\cos\beta)l_\beta - (1+\cos\beta)l_\alpha}{3 - 2\cos 2\beta}
\end{aligned}$$

The Lagrangian can now be rewritten as

$$\begin{aligned}
\mathcal{L}' &= \frac{1}{2}\left(\left((3+2\cos\beta)\dot{\alpha} + (1+\cos\beta)\dot{\beta}\right)\dot{\alpha} + \left((1+\cos\beta)\dot{\alpha} + \dot{\beta}\right)\dot{\beta}\right) + 2\cos\alpha + \cos(\alpha+\beta) \\
&= \frac{1}{2}l_\alpha\dot{\alpha} + \frac{1}{2}l_\beta\dot{\beta} + 2\cos\alpha + \cos(\alpha+\beta)
\end{aligned}$$

Using the definition of the Hamiltonian and our expressions for $\dot{\alpha}$ and $\dot{\beta}$, we can now get

$$\begin{aligned}
\mathcal{H} &\equiv l_\alpha\dot{\alpha} + l_\beta\dot{\beta} - \mathcal{L}' = \frac{1}{2}l_\alpha\dot{\alpha} + \frac{1}{2}l_\beta\dot{\beta} - 2\cos\alpha - \cos(\alpha+\beta) \\
&= \frac{1}{2}l_\alpha\left(2\frac{l_\alpha - (1+\cos\beta)l_\beta}{3-2\cos\beta}\right) + \frac{1}{2}l_\beta\left(2\frac{(3+2\cos\beta)l_\beta - (1+\cos\beta)l_\alpha}{3-2\cos\beta}\right) - 2\cos\alpha - \cos(\alpha+\beta) \\
&= \frac{l_\alpha^2 - 2(1+\cos\beta)l_\alpha l_\beta + (3+2\cos\beta)l_\beta^2}{3-2\cos\beta} - 2\cos\alpha - \cos(\alpha+\beta)
\end{aligned}$$

5.
Hand & Finch, p. 454, *Question* 12 (phase space flow equations)
Solution:
If the force (the *torque*, actually!) is $F\sin\omega t$, then from elementary considerations the equation of motion is

$$\begin{aligned}
ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta &= F\sin\omega t \\
\ddot{\theta} + \frac{b}{ml^2}\dot{\theta} + \frac{g}{l}\sin\theta &= \frac{F}{ml^2}\sin\omega t
\end{aligned}$$

Now define new constants: $\frac{1}{Q} \equiv \frac{b}{ml^2}$, $\omega_o^2 \equiv \frac{g}{l}$, and $f \equiv \frac{F}{ml^2}$. We then have:

$$\ddot{\theta} + \frac{\dot{\theta}}{Q} + \omega_o^2 sin\theta = f \sin\omega t$$

If we define $p \equiv \dot{\theta}$, then we can write

$$\begin{cases} \dot{\theta} = p \\ \dot{p} + \frac{p}{Q} + \omega_o^2 \sin\theta = f \sin\omega t \end{cases}$$

which is identical to Eq. 11.22.

6.
Hand & Finch, p. 455, *Question* 13 (symmetry breaking)
Solution:
If $t \to t + \frac{\pi}{\omega}$, $\frac{d}{dt} \to \frac{d}{dt}$ (derivatives are unaffected). So, with $\theta \to -\theta$ and $p \to -p$, eqn 11.22 becomes:

$$\begin{cases} -\dot{\theta} = -p \\ -\dot{p} = -\omega_o^2 \sin(-\theta) - \frac{-p}{Q} + f \sin(\omega(t + \frac{\pi}{\omega})) \end{cases}$$

$$\begin{cases} \dot{\theta} = p \\ \dot{p} = -\omega_o^2 \sin\theta - \frac{p}{Q} - f \sin(\omega t + \pi) \end{cases}$$

$$\begin{cases} \dot{\theta} = p \\ \dot{p} = -\omega_o^2 \sin\theta - \frac{p}{Q} + f \sin\omega t \end{cases}$$

which is identical to the original equations. Eqn 11.22 is thus invariant under this transformation.

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Physics 105 Fall 2000 Section 2 (*Strovink*)

PRACTICE EXAMINATION 1

Directions. Do all problems (weights are indicated). This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he will not give hints and will be obliged to write your question and its answer on the board. Roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (20 points)
In one dimension, the Lagrangian for a relativistic free electron is

$$\mathcal{L} = -mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}}\ ,$$

where m is the electron mass and c is the speed of light.

Find the total energy of the electron in terms of m, c, and $\dot{x}$. Prove your result given only the Lagrangian, using no other knowledge of relativity.

2. (25 points)
During $-\infty < t < 0$, a linear oscillator satisfying the equation of motion

$$\ddot{x} + \omega_0\dot{x} + \omega_0^2 x = \frac{F_x(t)}{m}$$

is driven at its resonant frequency by a force per unit mass

$$\frac{F_x(t)}{m} = a_0\cos\omega_0 t\ ,$$

where a_0 is a constant.
(**a**) (10 points)
Find $x(0)$ and $\dot{x}(0)$ at $t = 0$.
(**b**) (15 points)
At $t = 0$ the driving force is turned off. Find $x(t)$ for $t > 0$.

3. (35 points)
A small bead of mass m is constrained to move without friction on a circular hoop of radius a that rotates with constant angular velocity Ω about a vertical diameter. Use θ, the polar angle of the bead, as the single generalized coordinate ($\theta = 0$ at the bottom). Do not neglect gravity.
(**a**) (5 points)
Write the Lagrangian as a function of θ and $\dot{\theta}$. Remember to take into account the two different components of the bead's velocity.
(**b**) (5 points)
Obtain the differential equation of motion for θ.
(**c**) (10 points)
Find a restriction on Ω such that small oscillations about $\theta = 0$ can occur. What is the angular frequency of these oscillations?
(**d**) (5 points)
If Ω does not obey the restriction in part (**c**), about what other equilibrium position(s) can the bead undergo small oscillations?
(**e**) (10 points)
What is the angular frequency of the small oscillations to which part (**d**) refers?

4. (20 points)
An antiproton with mass m and charge $-e$ is incident upon a nucleus with mass M and charge Ze. You may assume them to be point particles. The nucleus is initially at rest. When m is still very far from M, it has velocity v_0, directed so that the two masses would miss by a distance b if there were no attraction between them. *State the system of units in which you are working (SI or cgs).*
(**a**) (10 points)
Obtain a pair of equations which, if solved, would allow you to calculate the distance of closest approach between the antiproton and the nucleus.

(**b**) (10 points)
As v_0 approaches zero, through what angle will the antiproton scatter? (Elementary reasoning, if stated correctly, should be sufficient here.)

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SOLUTION TO PRACTICE EXAMINATION 1

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1. (20 points)
In one dimension, the Lagrangian for a relativistic free electron is

$$\mathcal{L} = -mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}}\ ,$$

where m is the electron mass and c is the speed of light.

Find the total energy of the electron in terms of m, c, and $\dot{x}$. Prove your result given only this Lagrangian, using no other knowledge of relativity.
Solution:

$$\begin{aligned}
\mathcal{H} &\equiv \dot{x}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \mathcal{L} \\
&= \dot{x}\frac{-\frac{1}{2}mc^2}{\sqrt{1-\frac{\dot{x}^2}{c^2}}}\Big(\frac{-2\dot{x}}{c^2}\Big) + mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}} \\
&= \frac{mc^2}{\sqrt{1-\frac{\dot{x}^2}{c^2}}}\Big(\frac{\dot{x}^2}{c^2}+1-\frac{\dot{x}^2}{c^2}\Big) \\
&= \frac{mc^2}{\sqrt{1-\frac{\dot{x}^2}{c^2}}} \\
\frac{d\mathcal{H}}{dt} &= -\frac{\partial\mathcal{L}}{\partial t} = 0\ ,
\end{aligned}$$

so the Hamiltonian $\mathcal{H}$ is constant and equal to E, the total energy. Thus

$$E = \frac{mc^2}{\sqrt{1-\frac{\dot{x}^2}{c^2}}}\ .$$

2. (25 points)
During $-\infty < t < 0$, a linear oscillator satisfying the equation of motion

$$\ddot{x} + \omega_0\dot{x} + \omega_0^2 x = \frac{F_x(t)}{m}$$

is driven at its resonant frequency by a force per unit mass

$$\frac{F_x(t)}{m} = a_0\cos\omega_0 t\ ,$$

where a_0 is a constant.
(**a**) (10 points)
Find $x(0)$ and $\dot{x}(0)$ at $t = 0$.
Solution:
Since the driving force was first applied long ago at $t = -\infty$, the effects of that initial transient have died out and can be ignored; for $t < 0$ all we need is a particular solution. To get it, as usual we substitute

$$x = \mathrm{Re}\big(A\exp{(i\omega_0 t)}\big)$$

into the differential equation and choose to solve the complex version of the result, rather than its real part. Cancelling the common factor $\exp{(i\omega_0 t)}$, we obtain

$$\begin{aligned}
(-\omega_0^2 + i\omega_0^2 + \omega_0^2)A &= a_0 \\
A &= -\frac{ia_0}{\omega_0^2} \\
x(t<0) &= \frac{a_0}{\omega_0^2}\sin\omega_0 t \\
x(0) &= 0 \\
\dot{x}(0) &= \frac{a_0}{\omega_0}\ .
\end{aligned}$$

(**b**) (15 points)
At $t = 0$ the driving force is turned off. Find $x(t)$ for $t > 0$.
Solution:
Here we need a solution $x_h(t)$ to the homogeneous equation. Substituting

$$x_h = \mathrm{Re}\big(B \exp{(i\omega t)}\big)$$

with ω a constant to be determined, and cancelling the common factor $\exp{(i\omega t)}$, we obtain

$$\begin{aligned} 0 &= -\omega^2 + i\omega_0\omega + \omega_0^2 \\ \omega &= \frac{i\omega_0 \pm \sqrt{-\omega_0^2 + 4\omega_0^2}}{2} \\ &= -\frac{i\omega_0}{2} \pm \sqrt{\tfrac{3}{4}}\omega_0 \\ x_h(t) &= B \exp{(-\tfrac{1}{2}\omega_0 t)} \cos\Big(\sqrt{\tfrac{3}{4}}\omega_0 t + \beta\Big) \ , \end{aligned}$$

where B and β are adjustable constants. (This standard underdamped solution may also simply be recalled from memory or from notes.) Enforcing the initial condition $x(0) = 0$, we take $\beta = \frac{\pi}{2}$ and the solution becomes

$$x(t) = -B \exp{(-\tfrac{1}{2}\omega_0 t)} \sin\Big(\sqrt{\tfrac{3}{4}}\omega_0 t\Big) \ .$$

Matching the remaining initial condition,

$$\begin{aligned} \frac{a_0}{\omega_0} &= \dot{x}(0) \\ &= B\sqrt{\tfrac{3}{4}}\omega_0 \\ \sqrt{\tfrac{4}{3}}\frac{a_0}{\omega_0^2} &= B \\ x(t > 0) &= -\sqrt{\tfrac{4}{3}}\frac{a_0}{\omega_0^2} \exp{(-\tfrac{1}{2}\omega_0 t)} \sin\Big(\sqrt{\tfrac{3}{4}}\omega_0 t\Big) \ . \end{aligned}$$

3. (35 points)
A small bead of mass m is constrained to move without friction on a circular hoop of radius a that rotates with constant angular velocity Ω about a vertical diameter. Use θ, the polar angle of the bead, as the single generalized coordinate ($\theta = 0$ at the bottom). Do not neglect gravity.
(**a**) (5 points)
Write the Lagrangian as a function of θ and $\dot{\theta}$. Remember to take into account the two different components of the bead's velocity.
Solution:
The bead's velocity along the hoop ($a\dot{\theta}$) is orthogonal to the velocity associated with the hoop's rotation ($a \sin\theta\Omega$). From the center of the hoop, the height of the bead is $-a\cos\theta$. So the Lagrangian is

$$\begin{aligned} \mathcal{L} &= T - U \\ &= \tfrac{1}{2}ma^2(\dot{\theta}^2 + \Omega^2 \sin^2\theta) + mga\cos\theta \ . \end{aligned}$$

(**b**) (5 points)
Obtain the differential equation of motion for θ.
Solution:

$$\begin{aligned} \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= \frac{\partial\mathcal{L}}{\partial\theta} \\ \frac{d}{dt}(ma^2\dot{\theta}) &= ma^2\Omega^2 \sin\theta\cos\theta - mga\sin\theta \\ 0 &= \ddot{\theta} - \Omega^2\sin\theta\cos\theta + \frac{g}{a}\sin\theta \ . \end{aligned}$$

(**c**) (10 points)
Find a restriction on Ω such that small oscillations about $\theta = 0$ can occur. What is the angular frequency of these oscillations?
Solution:
To first order in $\theta \ll 1$, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. The equation of motion becomes

$$0 = \ddot{\theta} + \big(\frac{g}{a} - \Omega^2\big)\theta \ ,$$

which describes simple harmonic motion of θ with angular frequency

$$\omega_0 = \sqrt{\frac{g}{a} - \Omega^2} \ ,$$

provided that

$$\Omega < \sqrt{\frac{g}{a}} \ .$$

(**d**) (5 points)
If Ω does not obey the restriction in part (**c**), about what other equilibrium position(s) can the bead undergo small oscillations?

Solution:
At equilibrium, $\ddot{\theta} = 0$, so the equation of motion yields

$$0 = -\Omega^2 \sin\theta\cos\theta + \frac{g}{a}\sin\theta \ .$$

From the results of part (**c**), this equilibrium point occurs away from $\theta = 0$, so we can cancel the common factor $\sin\theta$. Then the equilibrium coordinate θ_0 becomes

$$\begin{aligned}\cos\theta_0 &= \frac{g}{a\Omega^2}\\ \theta_0 &= \pm\left|\arccos\left(\frac{g}{a\Omega^2}\right)\right| \ .\end{aligned}$$

(**e**) (10 points)
What is the angular frequency of the small oscillations to which part (**d**) refers?
Solution:
Applying the method of perturbations, substituting $\theta = \theta_0 + \psi$, we recalculate the angular factors to first order in ψ:

$$\begin{aligned}\theta &= \theta_0 + \psi\\ \sin\theta &\approx \sin\theta_0 + \psi\cos\theta_0\\ \cos\theta &\approx \cos\theta_0 - \psi\sin\theta_0\\ \sin\theta\cos\theta &\approx \sin\theta_0\cos\theta_0 + \psi(\cos^2\theta_0 - \sin^2\theta_0) \ .\end{aligned}$$

Applying these substitutions to the equation of motion,

$$\begin{aligned}0 &= \ddot{\theta} - \Omega^2\sin\theta\cos\theta + \frac{g}{a}\sin\theta\\ &\approx \ddot{\psi} - \Omega^2\big(\sin\theta_0\cos\theta_0 + \psi(\cos^2\theta_0 - \sin^2\theta_0)\big)\\ &+ \frac{g}{a}\big(\sin\theta_0 + \psi\cos\theta_0\big) \ .\end{aligned}$$

As usual, substituting $\cos\theta_0 = g/a\Omega^2$ from part (**d**) allows the terms independent of ψ to cancel:

$$0 = \ddot{\psi} - \Omega^2\psi(\cos^2\theta_0 - \sin^2\theta_0) + \frac{g}{a}\psi\cos\theta_0 \ .$$

The same substitution allows the terms containing $\cos\theta_0$ to cancel:

$$0 = \ddot{\psi} + \Omega^2\psi\sin^2\theta_0 \ .$$

This is an equation of simple harmonic motion for ψ with angular frequency

$$\begin{aligned}\omega_{\text{osc}} &= \Omega\sin\theta_0\\ &= \Omega\sqrt{1-\cos^2\theta_0}\\ &= \Omega\sqrt{1-\frac{g^2}{a^2\Omega^4}} \ .\end{aligned}$$

4. (20 points)
An antiproton with mass m and charge $-e$ is incident upon a nucleus with mass M and charge Ze. You may assume them to be point particles. The nucleus is initially at rest. When m is still very far from M, it has velocity v_0, directed so that the two masses would miss by a distance b if there were no attraction between them. *State the system of units in which you are working (SI or cgs).*
(**a**) (10 points)
Obtain a pair of equations which, if solved, would allow you to calculate the distance of closest approach between the antiproton and the nucleus.
Solution:
We shall work in SI. Initially the angular momentum of the two particles about their common center of mass is $\mu v_0 b$, where $\mu = mM/(m+M)$ is the reduced mass. At the perigee, when the particles are separated by a distance $r_{\min}$ and their relative velocity has magnitude $v_{\max}$, the angular momentum is the same because the force is central. Therefore

$$\begin{aligned}\mu v_{\max} r_{\min} &= \mu v_0 b\\ r_{\min} &= b\frac{v_0}{v_{\max}} \ .\end{aligned}$$

Initially, because the particles are greatly separated, their potential energy is zero. Therefore the initial energy has only a kinetic term, $\frac{1}{2}\mu v_0^2$. At the perigee, by energy conservation,

$$\tfrac{1}{2}\mu v_0^2 = \tfrac{1}{2}\mu v_{\max}^2 - \frac{Ze^2}{4\pi\epsilon_0 r_{\min}} \ .$$

This is a pair of equations that may be solved for the two unknowns $r_{\min}$ and $v_{\max}$.
(**b**) (10 points)
As v_0 approaches zero, through what angle will

the antiproton scatter? (Elementary reasoning, if stated correctly, should be sufficient here.)

Solution:

As $v_0 \to 0$, the total energy approaches zero as well. An orbit of zero total energy is a parabola, which has parallel asymptotes. Therefore the scattering angle will approach $180°$.

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EXAMINATION 1

Directions. Do both problems (weights are indicated). This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he will not give hints and will be obliged to write your question and its answer on the board. Roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (50 points)
A book has its front cover facing up (the normal to the cover is along $\hat{z}$). Its sentences are parallel to the $\hat{x}$ direction, and its binding is parallel to the $\hat{y}$ direction. Consider the "body" (x, y, z) axes to be attached to the book, with their origin at its CM. Define the "space" (x', y', z') axes initially to be the same as the (x, y, z) axes; however, the (x', y', z') axes are fixed – they don't change when the book rotates.

(**a**) (10 points)
Suppose the book is rotated about its z axis by 45° counterclockwise (carrying the body axes with it). The space axes remain fixed. Write down the transformation matrix Λ_a^t such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Lambda_a^t \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} .$$

(**b**) (10 points)
Instead suppose the book is rotated about its x axis by 45° counterclockwise (carrying the body axes with it). The space axes remain fixed. Write down the transformation matrix Λ_b^t such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Lambda_b^t \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} .$$

(**c**) (15 points)
Instead suppose the book is first rotated as in (**a**), next rotated as in (**b**). Write down the transformation matrix Λ_c^t such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Lambda_c^t \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} .$$

(**d**) (15 points)
Write down the inverse of Λ_c^t .

2. (50 points)
With respect to a fixed set of Cartesian coordinates (x, y, z), the position $\mathbf{r}(t)$ of a particle of mass m is given by

$$\mathbf{r}(t) = \hat{\mathbf{x}} x_0 + \hat{\mathbf{y}} v_0 t ,$$

where x_0 and v_0 are constants.

(**a**) (10 points)
With respect to the origin, write down the particle's moment of inertia $I(t)$.

(**b**) (15 points)
With respect to the origin, write down the magnitude and direction of the particle's *angular* velocity $\vec{\omega}(t)$.

(**c**) (10 points)
For the conditions specified in this problem, the product of $I(t)$ and $\vec{\omega}(t)$ is $\mathbf{L}$, the particle's angular momentum with respect to the origin. Write down $\mathbf{L}$. Is it a function of time t? If so, a torque must be acting on the particle – what is the source of this torque? Explain.

(**d**) (15 points)
Imagine that the particle in (**a**)-(**c**) is an element of fluid. The fluid's velocity field $\mathbf{v}(\mathbf{r})$ is given by

$$\mathbf{v}(\mathbf{r}) = \hat{\mathbf{y}} v_0 \frac{x}{x_0} ,$$

where, as above, x_0 and v_0 are constants. Can $\mathbf{v}(\mathbf{r})$ be expressed as the gradient of a scalar field $u(\mathbf{r})$, *i.e.*

$$\mathbf{v}(\mathbf{r}) = -\nabla u(\mathbf{r}) \ ?$$

If so, what is $u(\mathbf{r})$? If not, why not?

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SOLUTION TO EXAMINATION 1

Directions. Do both problems (weights are indicated). This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he will not give hints and will be obliged to write your question and its answer on the board. Roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (50 points)

A book has its front cover facing up (the normal to the cover is along $\hat{z}$). Its sentences are parallel to the $\hat{x}$ direction, and its binding is parallel to the $\hat{y}$ direction. Consider the "body" (x, y, z) axes to be attached to the book, with their origin at its CM. Define the "space" (x', y', z') axes initially to be the same as the (x, y, z) axes; however, the (x', y', z') axes are fixed – they don't change when the book rotates.

(**a**) (10 points)

Suppose the book is rotated about its z axis by 45° counterclockwise (carrying the body axes with it). The space axes remain fixed. Write down the transformation matrix Λ_a^t such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Lambda_a^t \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} .$$

Solution:

$$\Lambda_a^t = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} .$$

(**b**) (10 points)

Instead suppose the book is rotated about its x axis by 45° counterclockwise (carrying the body axes with it). The space axes remain fixed. Write down the transformation matrix Λ_b^t such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Lambda_b^t \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} .$$

Solution:

$$\Lambda_b^t = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} .$$

(**c**) (15 points)

Instead suppose the book is first rotated as in (**a**), next rotated as in (**b**). Write down the transformation matrix Λ_c^t such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Lambda_c^t \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} .$$

Solution:

$$\begin{aligned} \Lambda_c^t &= \Lambda_b^t \Lambda_a^t \\ &= \frac{1}{2} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \sqrt{2} & \sqrt{2} & 0 \\ -1 & 1 & \sqrt{2} \\ 1 & -1 & \sqrt{2} \end{pmatrix} . \end{aligned}$$

(**d**) (15 points)

Write down the inverse of Λ_c^t .

Solution: Λ_c^t is an orthogonal matrix, so its inverse is equal to its transpose:

$$\begin{aligned} (\Lambda_c^t)^{-1} &= \Lambda_c \\ &= \frac{1}{2} \begin{pmatrix} \sqrt{2} & -1 & 1 \\ \sqrt{2} & 1 & -1 \\ 0 & \sqrt{2} & \sqrt{2} \end{pmatrix} . \end{aligned}$$

2. (50 points)
With respect to a fixed set of Cartesian coordinates (x, y, z), the position $\mathbf{r}(t)$ of a particle of mass m is given by

$$\mathbf{r}(t) = \hat{\mathbf{x}} x_0 + \hat{\mathbf{y}} v_0 t \ ,$$

where x_0 and v_0 are constants.
(**a**) (10 points)
With respect to the origin, write down the particle's moment of inertia $I(t)$.
Solution:

$$I = mr^2 = m(x_0^2 + v_0^2 t^2) \ .$$

(**b**) (15 points)
With respect to the origin, write down the magnitude and direction of the particle's *angular* velocity $\vec{\omega}(t)$.
Solution:

$$\begin{aligned} \tan\varphi &= \frac{v_0 t}{x_0} \\ \frac{d\varphi}{\cos^2\varphi} &= \frac{v_0\, dt}{x_0} \\ \frac{d\varphi}{dt} &= \frac{v_0}{x_0}\cos^2\varphi \\ &= \frac{v_0}{x_0}\frac{x_0^2}{x_0^2 + v_0^2 t^2} \\ \vec{\omega} &= \hat{\mathbf{z}}\frac{v_0 x_0}{x_0^2 + v_0^2 t^2} \ . \end{aligned}$$

(**c**) (10 points)
For the conditions specified in this problem, the product of $I(t)$ and $\vec{\omega}(t)$ is $\mathbf{L}$, the particle's angular momentum with respect to the origin. Write down $\mathbf{L}$. Is it a function of time t? If so, a torque must be acting on the particle – what is the source of this torque? Explain.
Solution:

$$\begin{aligned} \mathbf{L} &= I\omega \\ &= m(x_0^2 + v_0^2 t^2)\, \hat{\mathbf{z}}\frac{v_0 x_0}{x_0^2 + v_0^2 t^2} \\ &= \hat{\mathbf{z}} m v_0 x_0 \ . \end{aligned}$$

$\mathbf{L}$ is a constant – its magnitude is just the momentum mv_0 times the impact parameter x_0, and its direction is the normal to the plane of motion. [This reasoning could be used to solve for $\vec{\omega}$ in (**b**), as an alternative to the solution given above].

(**d**) (15 points)
Imagine that the particle in (**a**)-(**c**) is an element of fluid. The fluid's velocity field $\mathbf{v}(\mathbf{r})$ is given by

$$\mathbf{v}(\mathbf{r}) = \hat{\mathbf{y}} v_0 \frac{x}{x_0} \ ,$$

where, as above, x_0 and v_0 are constants. Can $\mathbf{v}(\mathbf{r})$ be expressed as the gradient of a scalar field $u(\mathbf{r})$, *i.e.*

$$\mathbf{v}(\mathbf{r}) = -\nabla u(\mathbf{r}) \ ?$$

If so, what is $u(\mathbf{r})$? If not, why not?
Solution:
If $\mathbf{v}(\mathbf{r}) = -\nabla u(\mathbf{r})$, then $\nabla \times \mathbf{v} \equiv 0$. But

$$\nabla \times \mathbf{v} = \hat{\mathbf{z}}\frac{v_0}{x_0} \ ,$$

which is nonzero. Therefore $\mathbf{v}$ cannot be expressed as the gradient of a scalar field.

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EXAMINATION 2

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1. (60 points)
A block of mass m slides without friction on a wedge. The top surface of the wedge is at an angle α with respect to the horizontal. Denote by s the distance along the surface of the wedge between the mass and the tip of the wedge. Under the influence of gravity, $\ddot{s}$ will be negative (its value is not given).
(**a**) (5 points)
Write the potential energy $U(s)$ of the block as a function of s, defining $U(0) \equiv 0$.
(**b**) (5 points)
For this part, assume that the wedge is fixed. Write the kinetic energy T of the block as a function of $\dot{s}$.
(**c**) (10 points)
For the conditions of part (**b**), use s as the generalized coordinate and solve the Euler-Lagrange equation for $\ddot{s}$.
(**d**) (10 points)
For the remainder of this problem, assume that the wedge is allowed to slide without friction on a horizontal table. Take the wedge's horizontal coordinate to be x and its mass to be M. The block's coordinate s continues to be measured relative to the *wedge*, not the table. The kinetic energy of the block-wedge system becomes

$$T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\big(\dot{s}^2 + \dot{x}^2 - 2\dot{x}\dot{s}\cos\alpha\big) .$$

Write the Euler-Lagrange equation(s) using s and x as generalized coordinates.
(**e**) (15 points)
Find the cyclic coordinate and the conserved generalized momentum p that is canonically conjugate to it. If the block and the wedge are released from rest, what is the value of p?
(**f**) (15 points)
Using the non-cyclic Euler-Lagrange equation and the result of (**e**), solve for $\ddot{s}$. [*Hint:* If $\alpha = 30°$ and $M = m/2$, your answer should be twice what you found for (**c**).]

2. (40 points)
A mass m is connected to a fixed wall by a massless spring of constant k. The coordinate of the mass is $x = 0$ when the spring is relaxed. Neglect gravity.
(**a**) (10 points)
Assume that there is no damping force on m. At $t = 0$, m satisfies the initial conditions $x(0) = -x_0$, where x_0 is a positive constant, and $\dot{x}(0) = 0$. Write down $x(t)$ for $t > 0$.
(**b**) (10 points)
For the conditions of (**a**), denote the force of constraint exerted on the spring by the wall to be $F(t)$ (a function of time that is unknown until you figure it out). What is the earliest positive time $t_0 > 0$ at which $F(t_0)$ vanishes?
(**c**) (10 points)
Now assume that there exists an additional force $-b\dot{x}$ on m due to damping. For the initial conditions of (**a**), find the value approached by $x(t)$ in the limit $t \to \infty$.
(**d**) (10 points)
For the conditions of (**c**), it is observed that x never oscillates if the damping constant b is larger than a minimum value $b_{\min}$. What is $b_{\min}$? Explain the reasoning behind your answer.

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SOLUTION TO EXAMINATION 2

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(**a**) (5 points)
Write the potential energy $U(s)$ of the block as a function of s, defining $U(0) \equiv 0$.
Solution:

$$U = mgs\sin\alpha \;.$$

(**b**) (5 points)
For this part, assume that the wedge is fixed. Write the kinetic energy T of the block as a function of $\dot{s}$.
Solution:

$$T = \tfrac{1}{2}m\dot{s}^2 \;.$$

(**c**) (10 points)
For the conditions of part (**b**), use s as the generalized coordinate and solve the Euler-Lagrange equation for $\ddot{s}$.
Solution:

$$\begin{aligned}
\mathcal{L} &= \tfrac{1}{2}m\dot{s}^2 - mgs\sin\alpha \\
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{s}} &= \frac{\partial\mathcal{L}}{\partial s} \\
m\ddot{s} &= -mg\sin\alpha \\
\ddot{s} &= -g\sin\alpha \;.
\end{aligned}$$

(**d**) (10 points)
For the remainder of this problem, assume that the wedge is allowed to slide without friction on a horizontal table. Take the wedge's horizontal coordinate to be x and its mass to be M. The block's coordinate s continues to be measured relative to the *wedge*, not the table. The kinetic energy of the block-wedge system becomes

$$T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\big(\dot{s}^2 + \dot{x}^2 - 2\dot{x}\dot{s}\cos\alpha\big) \;.$$

Write the Euler-Lagrange equation(s) using s and x as generalized coordinates.
Solution:

$$\begin{aligned}
\mathcal{L} = {}& \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\big(\dot{s}^2 + \dot{x}^2 - 2\dot{x}\dot{s}\cos\alpha\big) \\
& - mgs\sin\alpha \;.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial s} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{s}} \\
-mg\sin\alpha &= \frac{d}{dt}\big(m\dot{s} - m\dot{x}\cos\alpha\big) \\
-mg\sin\alpha &= m\ddot{s} - m\ddot{x}\cos\alpha \;.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial x} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} \\
0 &= \frac{d}{dt}\big((M+m)\dot{x} - m\dot{s}\cos\alpha\big) \\
0 &= (M+m)\ddot{x} - m\ddot{s}\cos\alpha \;.
\end{aligned}$$

(**e**) (15 points)
Find the cyclic coordinate and the conserved generalized momentum p that is canonically conjugate to it. If the block and the wedge are released from rest, what is the value of p?
Solution:
The cyclic coordinate is x, because $\frac{\partial \mathcal{L}}{\partial x} = 0$. [Even if you do not remember what a cyclic coordinate is, you can see from the answer to (**d**) that

$$p = (M+m)\dot{x} - m\dot{s}\cos\alpha$$

is conserved; the generalized coordinate that is conjugate to this canonical momentum is x.] The system is released from rest, so $\dot{s}$ and $\dot{x}$ are zero at that time. Therefore, being conserved, $p = 0$ for all subsequent times.
(**f**) (15 points)
Using the non-cyclic Euler-Lagrange equation and the result of (**e**), solve for $\ddot{s}$. [*Hint:* If $\alpha = 30°$ and $M = m/2$, your answer should be twice what you found for (**c**).]
Solution: From the E-L equation in x,

$$\ddot{x} = \frac{m}{M+m}\ddot{s}\cos\alpha\ .$$

Substituting this into the E-L equation in s,

$$\ddot{s} - \frac{m}{M+m}\ddot{s}\cos^2\alpha = -g\sin\alpha$$
$$\ddot{s} = \frac{-g\sin\alpha}{1 - \frac{m}{M+m}\cos^2\alpha}\ .$$

2. (40 points)
A mass m is connected to a fixed wall by a massless spring of constant k. The coordinate of the mass is $x = 0$ when the spring is relaxed. Neglect gravity.
(**a**) (10 points)
Assume that there is no damping force on m. At $t = 0$, m satisfies the initial conditions $x(0) = -x_0$, where x_0 is a positive constant, and $\dot{x}(0) = 0$. Write down $x(t)$ for $t > 0$.
Solution:

$$x(t) = -x_0\cos\omega_0 t$$
$$\omega_0 \equiv \sqrt{\frac{k}{m}}\ .$$

(**b**) (10 points)
For the conditions of (**a**), denote the force of constraint exerted on the spring by the wall to be $F(t)$ (a function of time that is unknown until you figure it out). What is the earliest positive time $t_0 > 0$ at which $F(t_0)$ vanishes?
Solution:
The spring is massless, so there is no net force on it (otherwise it would have infinite acceleration). Therefore the force of the wall on the spring is equal to the force of the spring on the mass. The latter is just $m\ddot{x}$, which first vanishes at

$$\cos\omega_0 t_0 = \pi/2$$
$$t_0 = \frac{\pi}{2\omega_0}\ .$$

(**c**) (10 points)
Now assume that there exists an additional force $-b\dot{x}$ on m due to damping. For the initial conditions of (**a**), find the value approached by $x(t)$ in the limit $t \to \infty$.
Solution:
All homogeneous solutions (underdamped, overdamped, critically damped) contain a factor $\exp(-Ct)$ where C is some positive constant. Each possible solution will therefore vanish at $t \to \infty$, so $x(\infty) \to 0$.
(**d**) (10 points)
For the conditions of (**c**), it is observed that x never oscillates if the damping constant b is larger than a minimum value $b_{\min}$. What is $b_{\min}$? Explain the reasoning behind your answer.
Solution: The only homogeneous solution that oscillates (in addition to being damped) is the underdamped solution, which requires $Q > \frac{1}{2}$. Therefore, since there is no oscillation,

$$Q \le \tfrac{1}{2}$$
$$\frac{\omega_0}{\gamma} \le \tfrac{1}{2}$$
$$\frac{\omega_0}{b/m} \le \tfrac{1}{2}$$
$$2\omega_0 m \le b$$
$$2\sqrt{km} \le b\ .$$

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EXAMINATION 3

Directions. Do both problems (weights are indicated). This is a closed-book closed-note exam except for three $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he will not give hints and will be obliged to write your question and its answer on the board. Roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (50 points)
A satellite is in elliptical orbit about the earth (neglect any effects of the moon or sun). Its radius r is proportional to

$$r \propto \frac{1}{1+\epsilon\cos\theta} ,$$

where θ is the azimuthal angle of the orbit, and ϵ is the ellipse's *eccentricity*. For simplicity take $r_{\rm max} = 3r_{\rm min}$, so that $\epsilon = \frac{1}{2}$.

(**a**) (10 points)
Using any relevant theorem(s), write down the ratio $-\langle T\rangle/\langle U\rangle$, where T and U are the satellite's kinetic and potential energies, and $\langle\rangle$ is the time average over one full orbit.

(**b**) (20 points)
When $r = r_{\rm max}$, what is $-T/U$? [*Hint*: the satellite's total energy is inversely proportional to the semimajor axis of its orbit. If you don't remember the constant of proportionality, you can deduce it by considering the special case of a circular orbit.]

(**c**) (20 points)
When $r = r_{\rm max}$, a rocket on board the satellite fires a very brief burst, consuming fuel of negligible mass. Immediately after the burst, the satellite's total energy (normalized to zero at $r = \infty$) changes by a factor C, but its direction of motion remains the same; the satellite's orbit becomes *circular*. Solve for C.

2. (50 points)
When undriven, an undamped oscillator (*i.e.* a mass on a spring) satisfies the equation

$$\ddot{x} + \omega_0^2 x = 0 ,$$

where ω_0 is a positive constant. For $t < 0$ it is at rest at the origin: $x(t<0) = 0$.

(**a**) (20 points)
For this part, suppose that the mass is given a *quick tap* at $t = 0$, *i.e.*

$$x(t=0^+) = 0$$
$$\dot{x}(t=0^+) = v_0 ,$$

where v_0 is a positive constant. Solve for $x(t)$ for $t > 0$. [*Hint*: your solution should be equivalent to $v_0\,G(t)$, where $G(t)$ is the *Green function* for this oscillator.]

(**b**) (30 points)
For this part, suppose instead that the mass is given a *steady push* that begins at $t = 0$ and lasts for one period. That is, suppose that the force F on the mass, divided by the mass m, is such that $F/m = a(t)$, where

$$a(t) = a_0 \quad (0 < t < \tfrac{2\pi}{\omega_0})$$
$$= 0 \quad \text{otherwise} ,$$

where a_0 is a positive constant. Solve for $x(t)$ after the push is finished, *i.e.* for $t > 2\pi/\omega_0$.

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SOLUTION TO EXAMINATION 3

Directions. Do both problems (weights are indicated). This is a closed-book closed-note exam except for three $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he will not give hints and will be obliged to write your question and its answer on the board. Roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (50 points)
A satellite is in elliptical orbit about the earth (neglect any effects of the moon or sun). Its radius r is proportional to

$$r \propto \frac{1}{1+\epsilon\cos\theta} ,$$

where θ is the azimuthal angle of the orbit, and ϵ is the ellipse's *eccentricity*. For simplicity take $r_{\max} = 3r_{\min}$, so that $\epsilon = \frac{1}{2}$.
(**a**) (10 points)
Using any relevant theorem(s), write down the ratio $-\langle T\rangle/\langle U\rangle$, where T and U are the satellite's kinetic and potential energies, and $\langle\rangle$ is the time average over one full orbit.
Solution:
According to the Virial Theorem, if the (attractive) force varies as r^{-n},

$$\langle T\rangle = -\frac{n-1}{2}\langle U\rangle = -\tfrac{1}{2}\langle U\rangle .$$

(The above earns full credit. If you're lacking the details of the Virial Theorem, you need only recall that $-\langle T\rangle/\langle U\rangle$ depends only on the exponent of r in the force law. Consider a circular orbit in a $-k/r^2$ force field:

$$\begin{aligned} \frac{mv^2}{r} &= \frac{k}{r^2} \\ \tfrac{1}{2}mv^2 &= \tfrac{1}{2}\frac{k}{r} \\ T &= -\tfrac{1}{2}U . \end{aligned}$$

On average, this is true also for an elliptical orbit, since the force law is the same.)

(**b**) (20 points)
When $r = r_{\max}$, what is $-T/U$? [*Hint*: the satellite's total energy is inversely proportional to the semimajor axis of its orbit. If you don't remember the constant of proportionality, you can deduce it by considering the special case of a circular orbit.]
Solution:
Take a to be the ellipse's semimajor axis. It is related to $r_{\max}$ by

$$\begin{aligned} 2a &= r_{\min} + r_{\max} \\ &= r_{\max}\left(\tfrac{1}{3}+1\right) \\ a &= \tfrac{2}{3}r_{\max} . \end{aligned}$$

If the (attractive) force is k/r^2, the satellite's total energy is

$$E = -\frac{k}{2a} .$$

(Lacking the constant of proportionality $-k/2$, you may deduce it from the circular orbit considered in the solution to (**a**):

$$\begin{aligned} E &= T + U \\ &= -\tfrac{1}{2}U + U \\ &= \tfrac{1}{2}U \\ &= -\frac{k}{2r} , \end{aligned}$$

when a reduces to r in that special case.) Solving for the kinetic energy at $r = r_{\max}$,

$$\begin{aligned} T &= E - U \\ &= -\frac{k}{2a} - \frac{-k}{r_{\max}} \\ &= -\frac{k}{2a} - \frac{-k}{\frac{3}{2}a} \\ &= \frac{k}{6a} \\ \frac{T}{U} &= \frac{k/6a}{-2k/3a} \\ T &= -\tfrac{1}{4}U \ . \end{aligned}$$

(**c**) (20 points)
When $r = r_{\max}$, a rocket on board the satellite fires a very brief burst, consuming fuel of negligible mass. Immediately after the burst, the satellite's total energy (normalized to zero at $r = \infty$) changes by a factor C, but its direction of motion remains the same; the satellite's orbit becomes *circular*. Solve for C.
Solution:
Immediately after the rocket fires, the satellite is still at the same radius (otherwise it would undergo infinite acceleration in the limit that the burst duration vanishes). Therefore, since it is now in circular orbit,

$$E' = -\frac{k}{2r_{\max}} \ .$$

The original total energy was

$$E = -\frac{k}{2a} \ .$$

Their ratio is

$$C = \frac{E'}{E} = \frac{a}{r_{\max}} = \tfrac{2}{3} \ .$$

Note that the total energy is reduced *in magnitude* by the rocket burst. However the gain in total energy is *positive* because the total energy remains negative in sign.

2. (50 points)
When undriven, an undamped oscillator (*i.e.* a mass on a spring) satisfies the equation

$$\ddot{x} + \omega_0^2 x = 0 \ ,$$

where ω_0 is a positive constant. For $t < 0$ it is at rest at the origin: $x(t < 0) = 0$.
(**a**) (20 points)
For this part, suppose that the mass is given a *quick tap* at $t = 0$, *i.e.*

$$\begin{aligned} x(t = 0^+) &= 0 \\ \dot{x}(t = 0^+) &= v_0 \ , \end{aligned}$$

where v_0 is a positive constant. Solve for $x(t)$ for $t > 0$. [*Hint*: your solution should be equivalent to $v_0\, G(t)$, where $G(t)$ is the *Green function* for this oscillator.]
Solution:
The general solution is

$$x_h = B\cos\left(\omega_0 t + \beta\right) \ .$$

Applying the initial conditions at $t = 0$,

$$\begin{aligned} 0 &= x_h(0) \\ &= B\cos\beta \\ \Rightarrow \beta &= \tfrac{\pi}{2} \\ \Rightarrow x_h(t) &= -B\sin\omega_0 t \\ v_0 &= \dot{x}_h(0) \\ &= -B\omega_0 \\ \Rightarrow B &= -\frac{v_0}{\omega_0} \\ \Rightarrow x_h(t) &= \frac{v_0}{\omega_0}\sin\omega_0 t \ . \end{aligned}$$

Alternatively, taking advantage of the hint, you may obtain the same result as the limit of the Green function for the underdamped oscillator as $\gamma \to 0$.
(**b**) (30 points)
For this part, suppose instead that the mass is given a *steady push* that begins at $t = 0$ and lasts for one period. That is, suppose that the force F on the mass, divided by the mass m, is such that $F/m = a(t)$, where

$$\begin{aligned} a(t) &= a_0 \quad (0 < t < \tfrac{2\pi}{\omega_0}) \\ &= 0 \qquad \text{otherwise} \ , \end{aligned}$$

where a_0 is a positive constant. Solve for $x(t)$ after the push is finished, *i.e.* for $t > 2\pi/\omega_0$.

Solution:

Method 1. Simple argument.

During the push, irrespective of details that depend on initial conditions, the motion must be periodic with period $2\pi/\omega_0$. Therefore, after one period, the system must revert to its initial conditions $x = 0$, $\dot{x} = 0$. Given these conditions at $t = 2\pi/\omega_0$, after the driving force has vanished the mass must remain in the same conditions, *i.e.* at rest at the equilibrium position $x = 0$.

Method 2. Green function.

Using the solution $x_a(t)$ and the hint from (**a**), the Green function for this oscillator is

$$\begin{aligned}
G(t) &= \frac{x_a(t)}{v_0} \\
&= \frac{\sin\omega_0 t}{\omega_0} \quad (t > 0) \\
&= 0 \quad \text{otherwise} \\
G(t-t') &= \frac{\sin\omega_0(t-t')}{\omega_0} \quad (t > t') \\
&= 0 \quad \text{otherwise} .
\end{aligned}$$

The Green function yields an integral equation for $x(t)$:

$$\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} dt'\, a(t')\, G(t-t') \\
&= \int_{-\infty}^{t} dt'\, a(t')\, \frac{\sin\omega_0(t-t')}{\omega_0} \\
x\big(t > \tfrac{2\pi}{\omega_0}\big) &= \int_0^{2\pi/\omega_0} dt'\, a_0\, \frac{\sin\omega_0(t-t')}{\omega_0} .
\end{aligned}$$

For any value of $t > \frac{2\pi}{\omega_0}$, this is proportional to the integral of a sinusoidal function over one period, which must vanish. Therefore

$$x\big(t > \tfrac{2\pi}{\omega_0}\big) = 0 .$$

Method 3. Brute force solution of equations of motion.

During the push, the equation of motion is

$$\ddot{x} + \omega_0^2 x = a_0 .$$

The general solution is the sum of x_h and x_p, where

$$\begin{aligned}
x_h &= B\cos(\omega_0 t + \beta) \\
x_p &= \frac{a_0}{\omega_0^2} .
\end{aligned}$$

Applying the initial conditions at $t = 0$, ,

$$\begin{aligned}
0 &= \dot{x}_h(0) + \dot{x}_p(0) \\
&= -B\omega_0 \sin\beta \\
\Rightarrow \beta &= 0 \\
0 &= x_h(0) + x_p(0) \\
&= B\cos\beta + \frac{a_0}{\omega_0^2} \\
&= B + \frac{a_0}{\omega_0^2} \\
\Rightarrow B &= -\frac{a_0}{\omega_0^2} \\
x_h(t) + x_p(t) &= \frac{a_0}{\omega_0^2}(1 - \cos\omega_0 t) \\
x(t) &= \frac{a_0}{\omega_0^2}(1 - \cos\omega_0 t) .
\end{aligned}$$

From this solution we deduce that, at $t = \frac{2\pi}{\omega_0}$,

$$\begin{aligned}
x\big(t = \tfrac{2\pi}{\omega_0}\big) &= 0 \\
\dot{x}\big(t = \tfrac{2\pi}{\omega_0}\big) &= 0 .
\end{aligned}$$

After the push, the equation of motion is

$$\ddot{x} + \omega_0^2 x = 0 .$$

The general solution is

$$x_h = A\cos(\omega_0 t + \alpha) .$$

Applying the initial conditions at $t = \frac{2\pi}{\omega_0}$, *i.e.* the final conditions of the push,

$$\begin{aligned}
0 &= x\big(t = \tfrac{2\pi}{\omega_0}\big) \\
&= A\cos(2\pi + \alpha) \\
&= A\cos\alpha \\
\Rightarrow \alpha &= \tfrac{\pi}{2} \;\text{ or }\; A = 0 \\
0 &= \dot{x}\big(t = \tfrac{2\pi}{\omega_0}\big) \\
&= -A\omega_0 \sin(2\pi + \alpha) \\
\Rightarrow \alpha &= 0 \;\text{ or }\; A = 0 .
\end{aligned}$$

The only mutually consistent way to satisfy both boundary conditions is $A = 0$, so

$$x\big(t > \tfrac{2\pi}{\omega_0}\big) = 0 .$$

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EXAMINATION 4

Directions. Do both problems (weights are indicated). This is a closed-book closed-note exam except for four $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he will not give hints and will be obliged to write your question and its answer on the board. Roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (50 points)
Four thin rods of length a and mass m are welded together to form a square picture frame of side a and mass $4m$. In a set of body axes tied to the frame, the frame lies in the xy plane; its corner is at the origin and its sides lie along the $\hat{x}$ and $\hat{y}$ axes. (Note that the origin does not coincide with the frame's CM.)

The frame is rotating counterclockwise about the (body or space) $\hat{y}$ axis with uniform angular velocity ω. In the body axes, what are the components of its angular momentum?

2. (50 points)
Consider a football in a force-free environment. Neglecting its seams and laces, approximate it as cylindrically symmetric about its own $\hat{3}$ (body) axis. At $t = 0$, a set of space ($'$) axes is (momentarily) coincident with the body axes. As seen in the space axes at $t = 0$, the CM of the football is at rest at the origin; the football is rotating counterclockwise about the (space) $\hat{2}'$ axis with angular velocity ω (this is "end-over-end" rotation).

Give a qualitative description of the football's motion for $t > 0$ as seen in its own (body) system. Justify your assertions. Does the football's angular momentum appear to stay constant when it is observed in the body system?

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SOLUTION TO EXAMINATION 4

Directions. Do both problems (weights are indicated). This is a closed-book closed-note exam except for four $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he will not give hints and will be obliged to write your question and its answer on the board. Roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (50 points)
Four thin rods of length a and mass m are welded together to form a square picture frame of side a and mass $4m$. In a set of body axes tied to the frame, the frame lies in the xy plane; its corner is at the origin and its sides lie along the $\hat{x}$ and $\hat{y}$ axes. (Note that the origin does not coincide with the frame's CM.)

The frame is rotating counterclockwise about the (body or space) $\hat{y}$ axis with uniform angular velocity ω. In the body axes, what are the components of its angular momentum?

Solution:
The angular momentum is $\tilde{L} = \mathcal{I}\tilde{\omega}$. Displaying the components,

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} \mathcal{I}_{xx} & \mathcal{I}_{xy} & \mathcal{I}_{xz} \\ \mathcal{I}_{yx} & \mathcal{I}_{yy} & \mathcal{I}_{yz} \\ \mathcal{I}_{zx} & \mathcal{I}_{zy} & \mathcal{I}_{zz} \end{pmatrix} \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} = \begin{pmatrix} \mathcal{I}_{xy}\omega \\ \mathcal{I}_{yy}\omega \\ \mathcal{I}_{zy}\omega \end{pmatrix} .$$

$\mathcal{I}_{zy}$ vanishes because the frame is in the plane $z = 0$, so we are left with the task of calculating $\mathcal{I}_{xy}$ and $\mathcal{I}_{yy}$. Orient the picture frame so that "up" is along $\hat{y}$ and "to the right" is along $\hat{x}$, and denote the rods by "up", "down", "left", and "right". The diagonal component $\mathcal{I}_{yy}$ is the frame's scalar moment of inertia for rotation about the $\hat{y}$ axis. To it the right-hand rod contributes ma^2 and the left-hand rod contributes 0 (because it coincides with the $\hat{y}$ axis). The top and bottom rods each contribute

$$\frac{m}{a}\int_0^a (x^2 + z^2)\, dx = \tfrac{1}{3}ma^2 .$$

The total is

$$I_{yy} = ma^2 + 2(\tfrac{1}{3}ma^2) = \tfrac{5}{3}ma^2 .$$

As for the off-diagonal component $\mathcal{I}_{xy}$, it is proportional to xy integrated over the frame. For the left-hand rod $x = 0$ and for the bottom rod $y = 0$, so they don't contribute. The contribution of the top rod is

$$-\frac{m}{a}\int_0^a xy\, dx = -\tfrac{1}{2}ma^2 ;$$

the right-hand rod's contribution is the same. Then

$$I_{xy} = -\tfrac{1}{2}ma^2 - \tfrac{1}{2}ma^2 = -ma^2 .$$

Putting it together,

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = ma^2\omega \begin{pmatrix} -1 \\ \frac{5}{3} \\ 0 \end{pmatrix} .$$

2. (50 points)
Consider a football in a force-free environment. Neglecting its seams and laces, approximate it as cylindrically symmetric about its own $\hat{3}$ (body) axis. At $t = 0$, a set of space ($'$) axes is (momentarily) coincident with the body axes. As seen in the space axes at $t = 0$, the CM of the football is at rest at the origin; the football is rotating counterclockwise about the (space) $\hat{2}'$ axis with angular velocity ω (this is "end-over-end" rotation).

Give a qualitative description of the football's motion for $t > 0$ as seen in its own (body) system. Justify your assertions. Does the football's angular momentum appear to stay constant when it is observed in the body system?

Solution:

You may use its cylindrical symmetry to argue that the football's $\hat{2}$ axis a principal axis. Pure rotation about any principal axis always proceeds without wobbling, so $\vec{\omega}$ and $\mathbf{L}$ are constant (and nonzero!). This simple argument is sufficient to earn full credit.

Alternatively, you may use the Euler equations. In the body system,

$$\begin{aligned}
\mathcal{I}_{33}\dot{\omega}_3 - (\mathcal{I}_{11} - \mathcal{I}_{22})\omega_1\omega_2 &= N_3 \\
\mathcal{I}_{11}\dot{\omega}_1 - (\mathcal{I}_{22} - \mathcal{I}_{33})\omega_2\omega_3 &= N_1 \\
\mathcal{I}_{22}\dot{\omega}_2 - (\mathcal{I}_{33} - \mathcal{I}_{11})\omega_3\omega_1 &= N_2 .
\end{aligned}$$

In this force-free environment, all torque components N_i vanish. Since ω_2 is the only nonvanishing component of $\vec{\omega}$, all of the terms proportional to $\omega_i\omega_j$ also vanish. Therefore the time derivatives of all the components of $\vec{\omega}$ vanish, forcing $\vec{\omega}$ to remain constant. The angular momentum $\mathbf{L} = \hat{2}\mathcal{I}_{22}\omega$ likewise remains constant. Again this argument is sufficient for full credit.

Optionally, you may be worried about the stability of rotation about the $\hat{2}$ axis. According to lecture notes (**10.3**),

$$\frac{\ddot{\omega}_i}{\omega_i} \propto (I_k - I_j)(I_k - I_i) \ ,$$

where $\hat{k}$ is the main axis of rotation and $\hat{\imath}$ is an orthogonal direction along which a tiny stray component of $\vec{\omega}$ might grow. However, for the cylindrically symmetric football, two of the three principal moments of inertia are equal, so the right-hand term must vanish; correspondingly $\ddot{\omega}_i$ vanishes and the end-over-end rotation is stable. (You lose credit if you claim that this rotation is unstable, but you are not required to show that it is stable.)

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PRACTICE FINAL EXAMINATION

Directions. Do all six problems (weights are indicated). This is a closed-book closed-note exam except for two $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he will not give hints and will be obliged to write your question and its answer on the board. Roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (30 points)
A mass m is connected to a wall by a spring of constant k. Define $\omega_0 \equiv \sqrt{k/m}$. There is no damping force.
(**a**) (20 points)
In addition to the spring force, the mass is subjected to an external force F_{ext}:

$$\begin{aligned} F_{\text{ext}} &= 0, \quad t < 0 \\ &= F_0 \sin 2\omega_0 t, \quad 0 < t < 2\pi/\omega_0 \\ &= 0, \quad 2\pi/\omega_0 < t \ , \end{aligned}$$

where F_0 and ω_0 are constants. Find $x(t)$ for $t > 2\pi/\omega_0$.
(*Hint*: A solution to the differential equation

$$\Big(\frac{d^2}{dt^2} + \omega_0^2\Big) G(t) = \delta(t) \ ,$$

where $\delta(t)$ is a Dirac delta function, and G and $\dot{G}$ vanish for $t \le 0$, is

$$\begin{aligned} G(t) &= 0, \quad t \le 0 \\ &= \frac{\sin \omega_0 t}{\omega_0}, \quad t > 0 \ .) \end{aligned}$$

(**b**) (10 points)
As an alternative to applying an external force F_{ext}, this oscillator could be excited by causing the spring "constant" k to vary sinusoidally with time:

$$k(t) = k_0(1 + \epsilon_0 \cos \Omega t) \ ,$$

where ϵ_0 and Ω are constants. If such a variation were to occur for a long time, even if $\epsilon_0 \ll 1$, certain value(s) of Ω would cause the mass to oscillate with an amplitude that grows exponentially with time. Can you provide an example of such a value for Ω? (Here you are asked merely to recall a result from the reading and classroom discussion in which you have participated.)

2. (20 points)
A mass m is in uniform circular motion at constant radius R about a center of attractive force

$$\mathbf{F}(r) = -\frac{K\hat{r}}{r^2},$$

where K is a positive constant. The mass receives a slight nudge, causing the radius of its orbit to acquire a small perturbed component, so that

$$r(t) = R(1 + \epsilon \cos \omega' t),$$

where ϵ is a constant $\ll 1$. Find the angular frequency ω' of the perturbation. Justify your answer either by explicit calculation, or by simple arguments based on your knowledge of the orbit.

3. (35 points)
A square thin metal plate has area b^2 and mass m. A set of body axes is set up with the origin at the CM of the plate. $\hat{z}$ is normal to the plate, while $\hat{x}$ and $\hat{y}$ intersect the plate's *corners*. At $t = 0$ the angular velocity of the plate is

$$\vec{\omega}(0) = \frac{\omega_0}{\sqrt{2}}(\hat{x} + \hat{z}) \ .$$

(**a**) (15 points)
At $t = 0$, compute the angular momentum $\vec{L}(0)$ (measured in the body system).
(**b**) (15 points)
The motion of the plate is allowed to evolve freely, without the influence of any external forces or torques. At what time will $\vec{L}$, as measured in the body system, be directed within the $\hat{y}$-$\hat{z}$ plane rather than the $\hat{x}$-$\hat{z}$ plane?
(**c**) (5 points)
In the absence of external torques, angular momentum is conserved. Does this fact conflict with your answer to part (**b.**)? Explain.

4. (35 points)
(**a**) (7 points)
A compact disk ("CD") of mass m and radius b is suspended from its center by a strictly vertical wire of torsional constant γ. (Ignore the wire's mass and the CD's hole.) The disk remains horizontal and is free only to twist (with azimuthal angle φ) in the horizontal plane, such that the potential energy stored in the twisted wire is

$$U(\psi) = \tfrac{1}{2}\gamma\varphi^2.$$

Find the frequency ω_a of small oscillations of φ.
(**b**) (7 points)
The system is now made more complicated by the addition of a second identical CD, suspended from the first CD by a second identical torsion wire. Take ψ to be the azimuthal angle by which CD #2 is twisted – its full twist, not its twist relative to CD #1. Thus the net amount by which wire #2 is twisted is $(\psi - \varphi)$. Using φ and ψ as generalized coordinates, find the 2×2 symmetric matrix $\mathcal{M}$ such that the kinetic energy T is given by

$$T = \tfrac{1}{2}\begin{pmatrix}\dot{\varphi} & \dot{\psi}\end{pmatrix}\mathcal{M}\begin{pmatrix}\dot{\varphi}\\ \dot{\psi}\end{pmatrix} .$$

(**c**) (7 points)
For the same system, find the 2×2 symmetric matrix $\mathcal{K}$ such that the potential energy U is given by

$$U = \tfrac{1}{2}\begin{pmatrix}\varphi & \psi\end{pmatrix}\mathcal{K}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} .$$

(**d**) (7 points)
Obtain the natural angular frequencies of oscillation of this system.
(**e**) (7 points)
Describe the motion of the two CDs when the system has only one normal mode excited (you may choose any mode you wish). Your description should specify the amplitude of ψ relative to that of φ.

5. (40 points)
A *spherical top* of mass m under the influence of gravity with one point fixed is described by the usual Euler angles ϕ (= azimuth of the top's axis about the vertical), θ (= polar angle of the top's axis with respect to vertical), and ψ (= azimuth of the top about its axis). Gravity pulls down on the top's CM, which is a distance h from the (frictionless) pivot. The top's kinetic energy is given by

$$T = \tfrac{1}{2}I(\dot{\phi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi}\cos\theta) ,$$

where I is the top's moment of inertia about its symmetry axis, and also its moment of inertia about any axis which is perpendicular to the symmetry axis and which passes through the pivot. (The fact that there is only a single moment of inertia I is the reason that this top is called "spherical". Since I is measured about the pivot, not about the CM, the top itself is not spherically symmetric.)
(**a**) (20 points)
A generalized force of constraint Q_ϕ (actually a torque about the vertical axis) is applied to the top so that ϕ is constrained to be constant. For the initial conditions $\theta(0) \equiv \theta_0 \ll 1$, $\dot{\theta}(0) = 0$, and $\dot{\psi}(0) = \omega_0$, solve for $\theta(t)$ in the regime $\theta \ll 1$.
(**b**) (20 points)
For the conditions of (**a.**), find the generalized force of constraint $Q_\phi(t)$ which must be exerted upon the top to keep ϕ = constant.

6. (40 points)
Consider a long narrow rectangular membrane which, in equilibrium, lies in the x-y plane; $\hat{x}$ is its long direction and $\hat{y}$ is its short direction. The membrane's displacement normal to the x-y plane is denoted by $z(x, y, t)$. The membrane is clamped at its long edges $y = 0$ and $y = b$, so that

$$z(x, 0, t) = z(x, b, t) = 0\ .$$

We wish to investigate the propagation of traveling sinusoidal waves $z(x, y, t)$ in the long direction $\hat{x}$.
(**a**) (6 points)
The Lagrangian density $\mathcal{L}'$ (per unit area of membrane) is given by

$$\mathcal{L}'(z, \tfrac{\partial z}{\partial x}, \tfrac{\partial z}{\partial y}, \tfrac{\partial z}{\partial t}, x, y, t) =$$
$$= \tfrac{1}{2}\sigma\left(\tfrac{\partial z}{\partial t}\right)^2 - \tfrac{1}{2}\beta\Big(\left(\tfrac{\partial z}{\partial x}\right)^2 + \left(\tfrac{\partial z}{\partial y}\right)^2\Big)\ ,$$

where σ is the membrane's mass per unit area, and β is a positive constant that is inversely proportional to its elasticity. Apply the Euler-Lagange equations to this Lagrangian density to obtain a partial differential equation for $z(x, y, t)$.
(**b**) (6 points)
Search for a trial solution in the form

$$z(x, y, t) = Y(y)\cos\left(k_x x - \omega t\right)\ ,$$

where $Y(y)$ is a function only of y, and k_x and ω are constants which for the moment are unspecified. Plug this solution into the equation you obtained for (**a.**). Dividing through by $\cos\left(k_x x - \omega t\right)$, obtain an ordinary differential equation for $Y(y)$.
(**c**) (6 points)
Applying the boundary conditions $z(x, 0, t) = z(x, b, t) = 0$, identify and choose a (non null) solution for $Y(y)$ which has the most gradual dependence on y that is possible given these conditions.
(**d**) (6 points)
Returning to the equation you obtained for (**a.**), plug in your answer to (**c.**) to obtain an equation relating k_x and ω.
(**e**) (6 points)
What is the minimum frequency $\omega_{\min}$ of sinusoidal waves that can propagate in the $\hat{x}$ direction without attenuation?
(**f**) (5 points)
If $\omega = \sqrt{2}\omega_{\min}$, calculate the *phase* velocity with which sinusoidal waves propagate in the $\hat{x}$ direction.
(**g**) (5 points)
What is the *group* velocity of the waves described in (**e.**)?

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO PRACTICE FINAL EXAMINATION

Directions. Do all six problems (weights are indicated). This is a closed-book closed-note exam except for two $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he will not give hints and will be obliged to write your question and its answer on the board. Roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (30 points)
A mass m is connected to a wall by a spring of constant k. Define $\omega_0 \equiv \sqrt{k/m}$. There is no damping force.
(**a**) (20 points)
In addition to the spring force, the mass is subjected to an external force F_{ext}:

$$\begin{aligned} F_{\text{ext}} &= 0, \quad t < 0 \\ &= F_0 \sin 2\omega_0 t, \quad 0 < t < 2\pi/\omega_0 \\ &= 0, \quad 2\pi/\omega_0 < t \ , \end{aligned}$$

where F_0 and ω_0 are constants. Find $x(t)$ for $t > 2\pi/\omega_0$.
(*Hint*: A solution to the differential equation

$$\Big(\frac{d^2}{dt^2} + \omega_0^2\Big) G(t) = \delta(t) \ ,$$

where $\delta(t)$ is a Dirac delta function, and G and $\dot{G}$ vanish for $t \le 0$, is

$$\begin{aligned} G(t) &= 0, \quad t \le 0 \\ &= \frac{\sin \omega_0 t}{\omega_0}, \quad t > 0 \text{ .)} \end{aligned}$$

Solution:
The equation given for $G(t)$ defines it to be a Green function. Correspondingly, the solution for $x(t)$ is

$$x(t) = \int_{-\infty}^{t} \frac{F_{\text{ext}}(t')}{m} G(t - t')\, dt'$$

$$x(t > \tfrac{2\pi}{\omega_0}) = \int_0^{\frac{2\pi}{\omega_0}} \frac{F_0}{m} \sin 2\omega_0 t' \frac{\sin \omega_0 (t - t')}{\omega_0}\, dt'$$

$$= \int_0^{\frac{2\pi}{\omega_0}} \frac{F_0}{m} \sin 2\omega_0 t' \, \tfrac{\sin \omega_0 t \cos \omega_0 t' - \sin \omega_0 t' \cos \omega_0 t}{\omega_0}\, dt'$$

The integral in the last line splits into two pieces which are proportional, respectively, to

$$\int_0^{\frac{2\pi}{\omega_0}} \sin 2\omega_0 t' \cos \omega_0 t'\, dt' \quad \text{and}$$
$$\int_0^{\frac{2\pi}{\omega_0}} \sin 2\omega_0 t' \sin \omega_0 t'\, dt' \ .$$

The first piece vanishes because $\sin 2\omega_0 t'$ is odd and $\cos \omega_t'$ is even with respect to the midpoint of the interval; the second piece vanishes because $\sin my$ and $\sin ny$ are orthogonal functions when $m \neq n$. Therefore

$$x\big(t > \tfrac{2\pi}{\omega_0}\big) = 0 \ .$$

(**b**) (10 points)
As an alternative to applying an external force F_{ext}, this oscillator could be excited by causing the spring "constant" k to vary sinusoidally with time:

$$k(t) = k_0 (1 + \epsilon_0 \cos \Omega t) \ ,$$

where ϵ_0 and Ω are constants. If such a variation were to occur for a long time, even if $\epsilon_0 \ll 1$, certain value(s) of Ω would cause the mass to oscillate with an amplitude that grows exponentially with time. Can you provide an example of such a value for Ω? (Here you are asked merely to recall a result from the reading and classroom discussion in which you have participated.)

Solution:
This is the Mathieu equation describing *e.g.* a child pumping a swing. Its solution exhibits a parametric resonance at

$$\Omega = 2\omega_0 \ .$$

2. (20 points)
A mass m is in uniform circular motion at constant radius R about a center of attractive force

$$\mathbf{F}(r) = -\frac{K\hat{r}}{r^2},$$

where K is a positive constant. The mass receives a slight nudge, causing the radius of its orbit to acquire a small perturbed component, so that

$$r(t) = R(1 + \epsilon \cos \omega' t),$$

where ϵ is a constant $\ll 1$. Find the angular frequency ω' of the perturbation. Justify your answer either by explicit calculation, or by simple arguments based on your knowledge of the orbit.
Solution:
A $1/r^2$ attractive central force yields an elliptical bound orbit, with the focus of the ellipse located at the center of force. Relative to its focus, an ellipse has one fixed point of maximum radius and one fixed point of minimum radius. Therefore the angular frequency with which the radius varies is the same as the angular frequency ω_0 of the basic orbit. From centrifugal force balance for the (unperturbed) circular orbit, this is

$$\begin{aligned}
\frac{mv^2}{R} &= \frac{K}{R^2} \\
v &= \sqrt{\frac{K}{mR}} \\
\omega_0 &= \frac{v}{R} = \sqrt{\frac{K}{mR^3}} \ .
\end{aligned}$$

The same solution can be obtained with more effort by evaluating the effective spring constant

$$k_{\text{eff}} \equiv \left.\frac{d^2 U_{\text{eff}}}{dr^2}\right|_{\text{min}} ,$$

where the effective potential U_{eff} is the sum of $l^2/2mr^2$ and the potential that yields $\mathbf{F}(r)$; then $\omega = \sqrt{k_{\text{eff}}/m}$. As a second alternative, the method of perturbations may be applied.

3. (35 points)
A square thin metal plate has area b^2 and mass m. A set of body axes is set up with the origin at the CM of the plate. $\hat{z}$ is normal to the plate, while $\hat{x}$ and $\hat{y}$ intersect the plate's *corners*. At $t = 0$ the angular velocity of the plate is

$$\vec{\omega}(0) = \frac{\omega_0}{\sqrt{2}}(\hat{x} + \hat{z}) \ .$$

(**a**) (15 points)
At $t = 0$, compute the angular momentum $\vec{L}(0)$ (measured in the body system).
Solution:
First we need to compute the inertia tensor $\mathcal{I}$ of the plate. For the moment, imagine that the $\hat{x}$ and $\hat{y}$ axes pass through the *sides* rather than the *corners* of the plate. Then $\mathcal{I}_{xx}$ and $\mathcal{I}_{yy}$ would be easy to compute – either would be equal to $\frac{1}{12}mb^2$, the moment of inertia of a stick. By symmetry, $\hat{x}$ and $\hat{y}$ would be principal axes, as is $\hat{z}$. Since the (thin) plate is a plane,

$$\mathcal{I}_{zz} = \mathcal{I}_{xx} + \mathcal{I}_{yy} = \tfrac{1}{6}mb^2 \ .$$

How is the actual situation different, given that the $\hat{x}$ and $\hat{y}$ axes actually pass through the plate's corners? Not at all. Still, by symmetry, $\hat{x}$ and $\hat{y}$ are principal axes. The above equation still requires $\mathcal{I}_{xx}$ and $\mathcal{I}_{yy}$ to have the same value. Therefore

$$\mathcal{I} = \tfrac{1}{12}mb^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \ .$$

Finally

$$\begin{aligned}
\vec{L} &= \mathcal{I}\vec{\omega} \\
&= \tfrac{1}{12}mb^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \frac{\omega_0}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{mb^2\omega_0}{12\sqrt{2}}(\hat{x} + 2\hat{z}) \ .
\end{aligned}$$

(**b**) (15 points)
The motion of the plate is allowed to evolve freely, without the influence of any external forces or torques. At what time will $\vec{L}$, as measured in the body system, be directed within the $\hat{y}$-$\hat{z}$ plane rather than the $\hat{x}$-$\hat{z}$ plane?
Solution:
For ease of notation, define $\hat{x}$, $\hat{y}$, and $\hat{z}$ to be the "1", "2", and "3", directions, respectively. Euler's equations

$$\begin{aligned}\mathcal{I}_{33}\dot{\omega}_3 - (\mathcal{I}_{11} - \mathcal{I}_{22})\omega_1\omega_2 &= N_3\\ \mathcal{I}_{11}\dot{\omega}_1 - (\mathcal{I}_{22} - \mathcal{I}_{33})\omega_2\omega_3 &= N_1\\ \mathcal{I}_{22}\dot{\omega}_2 - (\mathcal{I}_{33} - \mathcal{I}_{11})\omega_3\omega_1 &= N_2\end{aligned}$$

become

$$\begin{aligned}\dot{\omega}_3 &= 0\\ \dot{\omega}_1 &= -\omega_2\omega_3\\ \dot{\omega}_2 &= +\omega_1\omega_3\ .\end{aligned}$$

Therefore ω_3 is a constant, and $\vec{\omega}_\perp$, the component of $\vec{\omega}$ that is $\perp$ to $\hat{z}$, rotates around $\hat{z}$ with angular velocity ω_3. So in $\frac{1}{4}$ of a period, or

$$\Delta t = \frac{\pi}{2\omega_3} = \frac{\pi}{\sqrt{2}\omega_0}\ ,$$

the angular velocity will rotate into the $\hat{x}$-$\hat{z}$ plane. By the results of (**a.**), so will the angular momentum.
(**c**) (5 points)
In the absence of external torques, angular momentum is conserved. Does this fact conflict with your answer to part (**b.**)? Explain.
Solution:
No contradiction is implied. Angular momentum is conserved only in an *inertial* system, where Newton's laws hold. In part (**b.**) we calculated the evolution of angular momentum in the *body* system, which is rotating and therefore not inertial.

4. (35 points)
(**a**) (7 points)
A compact disk ("CD") of mass m and radius b is suspended from its center by a strictly vertical wire of torsional constant γ. (Ignore the wire's mass and the CD's hole.) The disk remains horizontal and is free only to twist (with azimuthal angle φ) in the horizontal plane, such that the potential energy stored in the twisted wire is

$$U(\psi) = \tfrac{1}{2}\gamma\varphi^2.$$

Find the frequency ω_a of small oscillations of φ.
Solution:
The disk's moment of inertia is

$$I = \frac{m}{\pi b^2}\int_0^b r^2 2\pi r\, dr = \tfrac{1}{2}mb^2$$

(it can be recalled rather than calculated). The Lagrangian is

$$\mathcal{L} = \tfrac{1}{4}mb^2\dot{\varphi}^2 - \tfrac{1}{2}\gamma\varphi^2\ .$$

The Euler-Lagrange equation yields

$$\tfrac{1}{2}mb^2\ddot{\varphi} + \gamma\varphi = 0\ .$$

This is the usual equation for simple harmonic motion with angular frequency

$$\omega_a = \sqrt{\frac{2\gamma}{mb^2}}\ .$$

(**b**) (7 points)
The system is now made more complicated by the addition of a second identical CD, suspended from the first CD by a second identical torsion wire. Take ψ to be the azimuthal angle by which CD #2 is twisted – its full twist, not its twist relative to CD #1. Thus the net amount by which wire #2 is twisted is $(\psi - \varphi)$. Using φ and ψ as generalized coordinates, find the 2×2 symmetric matrix $\mathcal{M}$ such that the kinetic energy T is given by

$$T = \tfrac{1}{2}\begin{pmatrix}\dot{\varphi} & \dot{\psi}\end{pmatrix}\mathcal{M}\begin{pmatrix}\dot{\varphi}\\ \dot{\psi}\end{pmatrix}\ .$$

Solution:
In analogy to the single disk system, the kinetic energy is

$$\begin{aligned}T &= \tfrac{1}{4}mb^2\dot{\varphi}^2 + \tfrac{1}{4}mb^2\dot{\psi}^2\\ &= \tfrac{1}{4}mb^2\begin{pmatrix}\dot{\varphi} & \dot{\psi}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\dot{\varphi}\\ \dot{\psi}\end{pmatrix}\ .\end{aligned}$$

Thus

$$\mathcal{M} = \tfrac{1}{2}mb^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

(**c**) (7 points)
For the same system, find the 2×2 symmetric matrix $\mathcal{K}$ such that the potential energy U is given by

$$U = \tfrac{1}{2} \begin{pmatrix} \varphi & \psi \end{pmatrix} \mathcal{K} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} .$$

Solution:
In analogy to the single disk system, the potential energy is

$$\begin{aligned} U &= \tfrac{1}{2}\gamma\varphi^2 + \tfrac{1}{2}\gamma(\psi - \varphi)^2 \\ &= \tfrac{1}{2}\gamma(2\varphi^2 + \psi^2 - \varphi\psi - \psi\varphi) \\ &= \tfrac{1}{2}\gamma \begin{pmatrix} \varphi & \psi \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} . \end{aligned}$$

Thus

$$\mathcal{K} = \gamma \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} .$$

(**d**) (7 points)
Obtain the natural angular frequencies of oscillation of this system.
Solution:
The secular equation for the natural frequencies ω is

$$\begin{aligned} 0 &= \det(\mathcal{K} - \omega^2\mathcal{M}) \\ &= \left| \frac{2\gamma}{mb^2} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} - \omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| \\ &\equiv \left| \omega_a^2 \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} - \omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| . \end{aligned}$$

Defining $\eta \equiv \omega^2/\omega_a^2$,

$$\begin{aligned} 0 &= \det \begin{pmatrix} 2-\eta & -1 \\ -1 & 1-\eta \end{pmatrix} \\ &= 2 - 3\eta + \eta^2 - 1 \\ \eta &= \tfrac{3}{2} \pm \sqrt{\tfrac{9}{4} - 1} \\ &= \tfrac{3}{2} \pm \sqrt{\tfrac{5}{4}} \\ \omega^2 &= \tfrac{1}{2}\omega_a^2(3 \pm \sqrt{5}) \\ &= \frac{\gamma}{mb^2}(3 \pm \sqrt{5}) . \end{aligned}$$

(**e**) (7 points)
Describe the motion of the two CDs when the system has only one normal mode excited (you may choose any mode you wish). Your description should specify the amplitude of ψ relative to that of φ.
Solution:
The equation that yields the (unnormalized) eigenvectors is

$$0 = \begin{pmatrix} 2-\eta & -1 \\ -1 & 1-\eta \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} .$$

The top line of this pair of equations is

$$0 = (2-\eta)\varphi - \psi .$$

Substituting *e.g.* $\eta = \frac{1}{2}(3 - \sqrt{5})$ for the slower mode,

$$\psi = \frac{1+\sqrt{5}}{2}\varphi ,$$

so the lower CD twists in phase with the upper CD at $\approx 162\%$ of the upper CD's amplitude.

5. (40 points)
A *spherical top* of mass m under the influence of gravity with one point fixed is described by the usual Euler angles ϕ (= azimuth of the top's axis about the vertical), θ (= polar angle of the top's axis with respect to vertical), and ψ (= azimuth of the top about its axis). Gravity pulls down on the top's CM, which is a distance h from the (frictionless) pivot. The top's kinetic energy is given by

$$T = \tfrac{1}{2}I(\dot\phi^2 + \dot\theta^2 + \dot\psi^2 + 2\dot\phi\dot\psi\cos\theta) ,$$

where I is the top's moment of inertia about its symmetry axis, and also its moment of inertia about any axis which is perpendicular to the symmetry axis and which passes through the pivot. (The fact that there is only a single moment of inertia I is the reason that this top is called "spherical". Since I is measured about the pivot, not about the CM, the top itself is not spherically symmetric.)

(**a**) (20 points)
A generalized force of constraint Q_ϕ (actually a torque about the vertical axis) is applied to the top so that ϕ is constrained to be constant. For the initial conditions $\theta(0) \equiv \theta_0 \ll 1$, $\dot{\theta}(0) = 0$, and $\dot{\psi}(0) = \omega_0$, solve for $\theta(t)$ in the regime $\theta \ll 1$.

Solution:
The Lagrangian for the spherical top is

$$\mathcal{L} = \tfrac{1}{2}I(\dot{\phi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi}\cos\theta) - mgh\cos\theta \ .$$

The Euler-Lagrange equations are

$$\begin{aligned} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} &= \frac{\partial L}{\partial \phi} + Q_\phi \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= \frac{\partial L}{\partial \theta} \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} &= \frac{\partial L}{\partial \psi} \ . \end{aligned}$$

Applying these equations to the above Lagrangian,

$$\begin{aligned} I(\ddot{\phi} + \ddot{\psi}\cos\theta - \dot{\psi}\dot{\theta}\sin\theta) &= Q_\phi \\ I\ddot{\theta} &= (mgh - I\dot{\phi}\dot{\psi})\sin\theta \\ I(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta) &= 0 \ . \end{aligned}$$

Enforcing the constraint $\phi =$ constant, so that $\dot{\phi} = \ddot{\phi} = 0$, these equations simplify to

$$\begin{aligned} I(\ddot{\psi}\cos\theta - \dot{\psi}\dot{\theta}\sin\theta) &= Q_\phi \\ I\ddot{\theta} &= mgh\sin\theta \\ I\ddot{\psi} &= 0 \ . \end{aligned}$$

Only the second Euler-Lagrange equation is needed to solve this part of the problem. Approximating $\sin\theta \approx \theta$,

$$\begin{aligned} I\ddot{\theta} &= mgh\theta \\ \theta &= \theta_0 \cosh\sqrt{\tfrac{mgh}{I}}t \ . \end{aligned}$$

This is the familiar result for a falling stick; it is the same result as would be obtained if the top were not spinning at all.

(**b**) (20 points)
For the conditions of (**a.**), find the generalized force of constraint $Q_\phi(t)$ which must be exerted upon the top to keep $\phi =$ constant.

Solution:
From the third Euler-Lagrange equation, $\ddot{\psi}$ vanishes, so $\dot{\psi} = \omega_0$ for all time. The first Euler-Lagrange equation simplifies to

$$\begin{aligned} -I\omega_0\dot{\theta}\sin\theta &= Q_\phi \\ -I\omega_0\dot{\theta}\theta &\approx Q_\phi \ , \end{aligned}$$

where again we have made the small-angle approximation $\sin\theta \approx \theta$. Substituting $\theta(t)$ from part (**a.**),

$$\begin{aligned} &Q_\phi(t) = \\ &= -I\omega_0\theta_0^2\sqrt{\tfrac{mgh}{I}}\ \sinh\left(\sqrt{\tfrac{mgh}{I}}t\right)\ \cosh\left(\sqrt{\tfrac{mgh}{I}}t\right) \\ &= -\tfrac{1}{2}I\omega_0\theta_0^2\sqrt{\tfrac{mgh}{I}}\ \sinh\left(2\sqrt{\tfrac{mgh}{I}}t\right) \ . \end{aligned}$$

6. (40 points)
Consider a long narrow rectangular membrane which, in equilibrium, lies in the x-y plane; $\hat{x}$ is its long direction and $\hat{y}$ is its short direction. The membrane's displacement normal to the x-y plane is denoted by $z(x, y, t)$. The membrane is clamped at its long edges $y = 0$ and $y = b$, so that

$$z(x, 0, t) = z(x, b, t) = 0 \ .$$

We wish to investigate the propagation of traveling sinusoidal waves $z(x, y, t)$ in the long direction $\hat{x}$.

(**a**) (6 points)
The Lagrangian density $\mathcal{L}'$ (per unit area of membrane) is given by

$$\begin{aligned} &\mathcal{L}'(z, \tfrac{\partial z}{\partial x}, \tfrac{\partial z}{\partial y}, \tfrac{\partial z}{\partial t}, x, y, t) = \\ &= \tfrac{1}{2}\sigma\left(\tfrac{\partial z}{\partial t}\right)^2 - \tfrac{1}{2}\beta\left(\left(\tfrac{\partial z}{\partial x}\right)^2 + \left(\tfrac{\partial z}{\partial y}\right)^2\right) \ , \end{aligned}$$

where σ is the membrane's mass per unit area, and β is a positive constant that is inversely proportional to its elasticity. Apply the Euler-Lagange equations to this Lagrangian density to obtain a partial differential equation for $z(x, y, t)$.

Solution:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial(\frac{\partial z}{\partial t})}+\frac{d}{dx}\frac{\partial\mathcal{L}}{\partial(\frac{\partial z}{\partial x})}+\frac{d}{dy}\frac{\partial\mathcal{L}}{\partial(\frac{\partial z}{\partial y})}=\frac{\partial\mathcal{L}}{\partial z}$$
$$\sigma\frac{\partial^2 z}{\partial t^2}-\beta(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2})=0\ .$$

(**b**) (6 points)
Search for a trial solution in the form

$$z(x,y,t)=Y(y)\cos(k_x x-\omega t)\ ,$$

where $Y(y)$ is a function only of y, and k_x and ω are constants which for the moment are unspecified. Plug this solution into the equation you obtained for (**a.**). Dividing through by $\cos(k_x x-\omega t)$, obtain an ordinary differential equation for $Y(y)$.
Solution:

$$(-\sigma\omega^2 Y+\beta k_x^2 Y-\beta Y'')\cos(k_x x-\omega t)=0$$
$$\beta Y''+Y(\sigma\omega^2-\beta k_x^2)=0\ .$$

(**c**) (6 points)
Applying the boundary conditions $z(x,0,t)=z(x,b,t)=0$, identify and choose a (non null) solution for $Y(y)$ which has the most gradual dependence on y that is possible given these conditions.
Solution:
The general solution for Y will be a sum of $\sin\sqrt{\sigma\omega^2-\beta k_x^2}y$ and $\cos\sqrt{\sigma\omega^2-\beta k_x^2}y$ if $\sigma\omega^2>\beta k_x^2$, or a sum of $\sinh\sqrt{\beta k_x^2-\sigma\omega^2}y$ and $\cosh\sqrt{\beta k_x^2-\sigma\omega^2}y$ if $\sigma\omega^2<\beta k_x^2$. However, no sum of sinh and cosh can vanish at both $y=0$ and $y=b$. Therefore $\sigma\omega^2>\beta k_x^2$ and we have a sum of sin and cos. In order to vanish at $y=0$ and $y=b$, the cos part must vanish, and

$$Y(y)\propto\sin\frac{n\pi y}{b}\ ,$$

where n is a positive integer. The most gradual dependence on y occurs when $n=1$. Therefore

$$Y(y)\propto\sin\frac{\pi y}{b}\ .$$

(**d**) (6 points)
Returning to the equation you obtained for (**a.**), plug in your answer to (**c.**) to obtain an equation relating k_x and ω.
Solution:
Adopting the above solution for Y,

$$\frac{\beta\pi^2}{b^2}=\sigma\omega^2-\beta k_x^2$$
$$\omega^2=\frac{\beta}{\sigma}\Big(k_x^2+\frac{\pi^2}{b^2}\Big)\ .$$

(**e**) (6 points)
What is the minimum frequency $\omega_{\rm min}$ of sinusoidal waves that can propagate in the $\hat{x}$ direction without attenuation?
Solution:
For a sinusoidal wave to propagate in the $\hat{x}$ direction without attenuation, k_x must be real, so that $k_x^2>0$. Then from the previous equation

$$\omega_{\rm min}=\frac{\pi}{b}\sqrt{\frac{\beta}{\sigma}}\ .$$

(**f**) (5 points)
If $\omega=\sqrt{2}\omega_{\rm min}$, calculate the *phase* velocity with which sinusoidal waves propagate in the $\hat{x}$ direction.
Solution:
If $\omega=\sqrt{2}\omega_{\rm min}$, from (**d.**) $k_x^2=\pi^2/b^2$. Then

$$v_{\rm phase}=\frac{\omega}{k_x}=\frac{\sqrt{2}(\pi/b)\sqrt{\beta/\sigma}}{\pi/b}=\sqrt{\frac{2\beta}{\sigma}}\ .$$

(**g**) (5 points)
What is the *group* velocity of the waves described in (**e.**)?
Solution:

$$v_{\rm group}=\frac{d\omega}{dk_x}\ .$$

From (**d.**),

$$\omega\,d\omega=\frac{\beta}{\sigma}k_x\,dk_x\ .$$

Therefore

$$\begin{aligned} v_{\text{group}} &= \frac{\beta}{\sigma}\frac{k_x}{\omega} \\ &= \frac{\beta}{\sigma v_{\text{phase}}} \\ &= \sqrt{\frac{\beta}{2\sigma}}\ . \end{aligned}$$

Note that the geometric mean of v_{phase} and v_{group} remains equal to $\sqrt{\frac{\beta}{\sigma}}$, the "pure" phase velocity of waves traveling on the membrane when no boundary restrictions are applied. (This problem is simpler than, but similar to, the problem of EM wave propagation in a hollow rectangular waveguide.)

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

FINAL EXAMINATION

Directions. Do all six problems (weights are indicated). This is a closed-book closed-note exam except for five $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he will not give hints and will be obliged to write your question and its answer on the board. Roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (20 points)
In the northern hemisphere at colatitude λ (as measured from the north pole, equivalent to north latitude $\frac{\pi}{2} - \lambda$), an ice rink is built by pouring water into an enclosure and then allowing it to freeze. If the rink is built this way, an isolated hockey puck that lies at rest on the ice won't move at all, even if the ice is frictionless (which is the case here).

The puck (of mass m) is tied to a frictionless center swivel using a taut massless rope of length R. The puck is set into counterclockwise uniform circular motion about the swivel point. As seen by an observer standing on the ice, the puck has a constant angular velocity ω about the swivel, *i.e.* it retraces its path around the ice every $\frac{2\pi}{\omega}$ seconds. The puck moves *slowly*: you may *not* assume that ω is much larger than Ω, the angular frequency ($= 2\pi$/day) of the earth's rotation about its axis.

What is the tension τ in the rope? You may work this problem either in the rest frame of the observer, or in the rest frame of the puck – but you must *state which frame you are using*.

2. (30 points)
A fixed upright solid cone with a height h and a circular base of radius R has a frictionless surface. The cone intercepts a vertical rain of tiny hailstones, which scatter elastically off the curved part of the cone. Since they are so tiny, only a negligible fraction of the hailstones hit the very tip of the cone. Neglect gravity.
(**a**) (10 points)
Show that the scattering angle Θ of the hailstones is 2α, where $\alpha = \arctan(R/h)$ is the half-angle of the cone. You may use this result in the remainder of the problem.
(**b**) (10 points)
Using a purely geometrical argument, write down the total cross section σ_T for elastic scattering of a hailstone by the cone.
(**c**) (10 points)
Taking ϕ to be the hailstone's angle about the cone's azimuth, write down the differential cross section

$$\frac{d^2\sigma}{\sin\Theta\, d\Theta\, d\phi}$$

for elastic scattering of a hailstone by the cone. [*Hint*: integrating the differential cross section over the full solid angle should yield σ_T.]

3. (40 points)
A physical system has a Lagrangian that is normalized (scaled) to be dimensionless. It is equal to

$$\mathcal{L}(a, b, \dot{a}, \dot{b}, t) = \tfrac{1}{2}\dot{a}^2 + \tfrac{1}{2}(a\dot{b})^2 - a^n ,$$

where a and b are dimensionless generalized coordinates, $a > 0$, n is an unspecified integer, and the time t is normalized so that it is dimensionless as well.
(**a**) (10 points)
Use one Euler-Lagrange equation to find the conserved canonical momentum p_0 in terms of a, b, $\dot{a}$, and $\dot{b}$.
(**b**) (10 points)
Write the other Euler-Lagrange equation. Substitute p_0 so that this equation is expressed

entirely in terms of one of the generalized coordinates, its time derivatives, and constants.
(**c**) (10 points)
Find a condition on n such that it is possible for the surviving generalized coordinate in part (**b**) to be constant.
(**d**) (10 points)
If the "constant" generalized coordinate in part (**c**) were perturbed slightly, would it oscillate stably about its "constant" value? Explain.

4. (45 points)
A double pendulum consists of a top bob of mass $3m$, hung from the ceiling by a string of length ℓ; and a bottom bob of mass m, hung from the top bob by another string of length ℓ. The top string makes an angle $\phi \ll 1$ from the vertical direction; the bottom string makes an angle $\theta \ll 1$, also measured from the vertical direction. Note that the two bobs have different masses.
(**a**) (10 points)
If the kinetic energy is expressed in units of $m\ell^2$, the potential energy is expressed in units of $mg\ell$, and the time is expressed in units of $\sqrt{\ell/g}$, making all possible small-angle approximations, show that (within an additive constant) the Lagrangian can be written

$$\mathcal{L} = \tfrac{1}{2}\left(4\dot{\phi}^2 + 2\dot{\phi}\dot{\theta} + \dot{\theta}^2 - 4\phi^2 - \theta^2\right) .$$

You may use this result for the remainder of the problem.
(**b**) (15 points)
Expressed as a ratio to $\sqrt{g/\ell}$, find the angular frequencies of this system's normal modes. [*Hint*: the winding number of this system turns out to be $\sqrt{3}$.]
(**c**) (10 points)
If you were unable to make the approximations $\phi \ll 1$ and $\theta \ll 1$, you would need to solve this system numerically. This can be easier if you use a set of first-order coupled partial differential equations, rather than a set of second-order partial differential equations. Given the Lagrangian, how would you obtain this first-order set of equations? (Only an explanation of what you would do is required.)
(**d**) (10 points)
Again for the conditions of part (**c**) (ϕ and θ not necessarily small), assume that you have an ideal double pendulum, an arbitrarily fast and precise computer, and "fairly accurate" initial conditions. Would you expect to obtain a "fairly accurate" prediction for its motion? Would your expecations depend on the range of motion that is considered? Explain.

5. (40 points)
A physical system is described by a single dimensionless generalized coordinate $b(s,t)$ that is a function of two independent variables: a time variable t and a (one-dimensional) field variable s. When s and t are normalized (scaled) to be dimensionless, and the Lagrangian density $\mathcal{L}'$ is similarly renormalized, $\mathcal{L}'$ takes the form

$$\mathcal{L}'(b, \tfrac{\partial b}{\partial s}, \tfrac{\partial b}{\partial t}, s, t) = \tfrac{1}{2}\left(\tfrac{\partial b}{\partial s}\right)^2 - \tfrac{1}{2}\left(\tfrac{\partial b}{\partial t}\right)^2 .$$

(**a**) (10 points)
Using the version of the Euler-Lagrange equation that is appropriate for a Lagrangian density, show that the equation controlling the evolution of $b(s,t)$ is

$$\tfrac{\partial^2 b}{\partial s^2} - \tfrac{\partial^2 b}{\partial t^2} = 0 .$$

You may use this result in the remainder of this problem.
(**b**) (10 points)
If $-\infty < s < \infty$, *i.e.* there are no boundaries for s, what is the *general* solution $b(s,t)$ to this equation?
(**c**) (10 points)
Now impose the boundary condition

$$b(s=0,t) = b(s=1,t) = 0 .$$

What are the angular frequencies of the normal modes of this system?
(**d**) (10 points)
Finally, retaining the boundary condition introduced in part (**c**), impose the initial conditions

$$b(s,t=0) = \sin \pi s$$
$$\tfrac{\partial b}{\partial t}(s,t=0) = 0 .$$

What is the earliest time t_0 such that

$$b(s,t_0) = -b(s,t=0) ,$$

i.e. the field $b(s,t)$ reverses sign but is otherwise unchanged? Explain your reasoning.

6. (25 points)
A one-dimensional physical system with generalized coordinate q and canonically conjugate momentum p is described by a Hamiltonian $\mathcal{H}(q,p,t)$ that is a smooth function of the variables upon which it depends. This is a conservative system (no dissipation), so that $d\mathcal{H}/dt$ vanishes, *i.e.* $\mathcal{H}$ is a constant of the motion.
(**a**) (10 points)
Prove that $\partial\mathcal{H}/\partial t$ vanishes.
(**b**) (15 points)
This system also is characterized by a different smooth function $F(q,p,t)$ of the same variables. It is known that F is also a constant of the motion. The quantity $\partial F/\partial t$ describes the *explicit* time dependence of the function F; it can be nonzero even when F is conserved. Prove that $\partial F/\partial t$ is a constant of the motion.

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO FINAL EXAMINATION

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1. (20 points)
In the northern hemisphere at colatitude λ (as measured from the north pole, equivalent to north latitude $\frac{\pi}{2} - \lambda$), an ice rink is built by pouring water into an enclosure and then allowing it to freeze. If the rink is built this way, an isolated hockey puck that lies at rest on the ice won't move at all, even if the ice is frictionless (which is the case here).

The puck (of mass m) is tied to a frictionless center swivel using a taut massless rope of length R. The puck is set into counterclockwise uniform circular motion about the swivel point. As seen by an observer standing on the ice, the puck has a constant angular velocity ω about the swivel, *i.e.* it retraces its path around the ice every $\frac{2\pi}{\omega}$ seconds. The puck moves *slowly*: you may *not* assume that ω is much larger than Ω, the angular frequency ($= 2\pi$/day) of the earth's rotation about its axis.

What is the tension τ in the rope? You may work this problem either in the rest frame of the observer, or in the rest frame of the puck – but you must *state which frame you are using*.

Solution:
In the observer's frame, the puck is accelerating inward toward the swivel with acceleration $R\omega^2$, and it feels an outward Coriolis force equal to $2mR\Omega\cos\lambda$. Therefore

$$\tau = mR\big(\omega^2 + 2\omega\Omega\cos\lambda\big) .$$

In the puck's (*) frame, the vertical component of its angular velocity is $\omega^* = \omega + \Omega\cos\lambda$. This would seem to require a centrifugal force

$$\begin{aligned}\tau &= mR\omega^{*2} \\ &= mR\big(\omega^2 + 2\omega\Omega\cos\lambda + \Omega^2\cos^2\lambda\big) .\end{aligned}$$

However, the last term must be compensated by the normal force of the ice, since the tension must vanish if the puck isn't moving with respect to the ice ($\omega = 0$). Therefore

$$\tau = mR\big(\omega^2 + 2\omega\Omega\cos\lambda\big) .$$

2. (30 points)
A fixed upright solid cone with a height h and a circular base of radius R has a frictionless surface. The cone intercepts a vertical rain of tiny hailstones, which scatter elastically off the curved part of the cone. Since they are so tiny, only a negligible fraction of the hailstones hit the very tip of the cone. Neglect gravity.
(**a**) (10 points)
Show that the scattering angle Θ of the hailstones is 2α, where $\alpha = \arctan(R/h)$ is the half-angle of the cone. You may use this result in the remainder of the problem.
Solution:
For scattering off the cone, the hailstone's angle of incidence θ_{inc} must equal its angle of reflection θ_{refl}, because the frictionless surface of the cone can exert only a normal impulse on the hailstone. Since the angle of incidence is

$$\theta_{\text{inc}} = \frac{\pi}{2} - \alpha ,$$

the scattering angle is

$$\begin{aligned}\Theta &= \pi - \theta_{\rm inc} - \theta_{\rm refl} \\ &= \pi - 2(\tfrac{\pi}{2} - \alpha) \\ &= 2\alpha\ .\end{aligned}$$

(**b**) (10 points)
Using a purely geometrical argument, write down the total cross section σ_T for elastic scattering of a hailstone by the cone.
Solution:
As seen by the hailstones, the cone has a cross-sectional area equal to πR^2. Therefore

$$\sigma_T = \pi R^2\ .$$

(**c**) (10 points)
Taking ϕ to be the hailstone's angle about the cone's azimuth, write down the differential cross section

$$\frac{d^2\sigma}{\sin\Theta\, d\Theta\, d\phi}$$

for elastic scattering of a hailstone by the cone. [*Hint*: integrating the differential cross section over the full solid angle should yield σ_T.]
Solution:
From (**a**), the hailstones have $\Theta = 2\alpha$. Therefore the differential cross section is proportional to $\delta(\Theta - 2\alpha)$. Take the (unknown) constant of proportionality to be equal to C. Using the hint,

$$\begin{aligned}\sigma_T &= \int_0^\pi \sin\Theta\, d\Theta \int_0^{2\pi} d\phi\, \frac{d^2\sigma}{\sin\Theta\, d\Theta\, d\phi} \\ \pi R^2 &= \int_0^\pi \sin\Theta\, d\Theta \int_0^{2\pi} d\phi\, C\delta(\Theta - 2\alpha) \\ &= \int_0^\pi \sin\Theta\, d\Theta\, 2\pi\, C\delta(\Theta - 2\alpha) \\ &= 2\pi C \sin 2\alpha \\ C &= \frac{\pi R^2}{2\pi \sin 2\alpha} \\ \frac{d^2\sigma}{\sin\Theta\, d\Theta\, d\phi} &= \frac{R^2}{2\sin 2\alpha}\delta(\Theta - 2\alpha)\ .\end{aligned}$$

3. (40 points)
A physical system has a Lagrangian that is normalized (scaled) to be dimensionless. It is equal to

$$\mathcal{L}(a, b, \dot a, \dot b, t) = \tfrac{1}{2}\dot a^2 + \tfrac{1}{2}(a\dot b)^2 - a^n\ ,$$

where a and b are dimensionless generalized coordinates, $a > 0$, n is an unspecified integer, and the time t is normalized so that it is dimensionless as well.
(**a**) (10 points)
Use one Euler-Lagrange equation to find the conserved canonical momentum p_0 in terms of a, b, $\dot a$, and $\dot b$.
Solution:

$$\begin{aligned}0 &= \frac{\partial \mathcal{L}}{\partial b} \\ \text{constant} &= \frac{\partial \mathcal{L}}{\partial \dot b} \\ p_0 &= a^2 \dot b\ .\end{aligned}$$

(**b**) (10 points)
Write the other Euler-Lagrange equation. Substitute p_0 so that this equation is expressed entirely in terms of one of the generalized coordinates, its time derivatives, and constants.
Solution:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial a} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot a} \\ -na^{n-1} + a\dot b^2 &= \ddot a \\ -na^{n-1} + a\left(\frac{p_0}{a^2}\right)^2 &= \ddot a \\ -na^{n-1} + \frac{p_0^2}{a^3} &= \ddot a\ .\end{aligned}$$

(**c**) (10 points)
Find a condition on n such that it is possible for the surviving generalized coordinate in part (**b**) to be constant.
Solution:

$$\begin{aligned}-na^{n-1} + \frac{p_0^2}{a_0^3} &= 0 \\ \frac{p_0^2}{a_0^3} &= na_0^{n-1} \\ n &> 0\ .\end{aligned}$$

(**d**) (10 points)

If the "constant" generalized coordinate in part **(c)** were perturbed slightly, would it oscillate stably about its "constant" value? Explain.

Solution:

Examining the structure of the Lagrangian, the actual (normalized) potential is a^n and the pseudopotential is $\frac{1}{2}(a\dot{b})^2 = \frac{1}{2}p_0^2/a^2$. The effective potential, which is the sum of the two, is equal to $+\infty$ both at $a = 0$ and at $a = \infty$. Thus the extremum in the effective potential, where it is possible for a to be constant, is a minimum rather than a maximum. Therefore a is stable about its constant value.

Alternatively, the effective potential may be differentiated twice to get the effective spring constant, or the method of perturbations may be applied.

4. (45 points)

A double pendulum consists of a top bob of mass $3m$, hung from the ceiling by a string of length ℓ; and a bottom bob of mass m, hung from the top bob by another string of length ℓ. The top string makes an angle $\phi \ll 1$ from the vertical direction; the bottom string makes an angle $\theta \ll 1$, also measured from the vertical direction. Note that the two bobs have different masses.

(**a**) (10 points)

If the kinetic energy is expressed in units of $m\ell^2$, the potential energy is expressed in units of $mg\ell$, and the time is expressed in units of $\sqrt{\ell/g}$, making all possible small-angle approximations, show that (within an additive constant) the Lagrangian can be written

$$\mathcal{L} = \tfrac{1}{2}\big(4\dot\phi^2 + 2\dot\phi\dot\theta + \dot\theta^2 - 4\phi^2 - \theta^2\big)\ .$$

You may use this result for the remainder of the problem.

Solution:

Let (x, y) be the coordinates of the top bob, and (u, v) be the coordinates of the bottom bob, with respect to their respective equilibrium points. Then

$$\begin{aligned} x &= \ell\sin\phi \approx \ell\phi \\ y &= \ell(1-\cos\phi) \approx \tfrac{1}{2}\ell\phi^2 \\ u &= x + \ell\sin\theta \approx x + \ell\theta \\ v &= u + \ell(1-\cos\theta) \approx y + \tfrac{1}{2}\ell\theta^2\ . \end{aligned}$$

The kinetic energy is

$$\begin{aligned} \frac{2T}{m\ell^2} &= 3(\dot x^2 + \dot y^2) + \dot u^2 + \dot v^2 \\ &\approx 3\dot x^2 + \dot u^2 \\ &= 3\dot\phi^2 + (\dot\phi + \dot\theta)^2 \\ &= 4\dot\phi^2 + 2\dot\phi\dot\theta + \dot\theta^2\ . \end{aligned}$$

The potential energy is

$$\begin{aligned} \frac{2U}{mg\ell} &= 3y + v \\ &= 3\phi^2 + (\phi^2 + \theta^2) \\ &= 4\phi^2 + \theta^2\ . \end{aligned}$$

Therefore the normalized Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}\big(4\dot\phi^2 + 2\dot\phi\dot\theta + \dot\theta^2 - 4\phi^2 - \theta^2\big)\ .$$

(**b**) (15 points)

Expressed as a ratio to $\sqrt{g/\ell}$, find the angular frequencies of this system's normal modes. [*Hint*: the winding number of this system turns out to be $\sqrt{3}$.]

Solution:

Applying the Euler-Lagrange equations yields

$$\begin{aligned} -4\phi &= 4\ddot\phi + \ddot\theta \\ -\theta &= \ddot\theta + \ddot\phi\ . \end{aligned}$$

Looking for solutions of the form

$$\begin{aligned} \phi &= \phi_0\cos\omega t \\ \theta &= \phi_0\cos\omega t\ , \end{aligned}$$

we obtain

$$\begin{aligned} 0 &= 4(1-\omega^2)\phi_0 - \omega^2\theta_0 \\ 0 &= (1-\omega^2)\theta_0 - \omega^2\phi_0\ . \end{aligned}$$

The secular determinant must vanish:

$$\begin{aligned} 0 &= 4(1-\omega^2)^2 - \omega^2 \\ &= 3\omega^4 - 8\omega^2 + 4 \\ \omega^2 &= \frac{3 \pm 2}{6} \\ \omega &= \sqrt{\frac{2}{3}} \text{ or } \sqrt{2}\ . \end{aligned}$$

(**c**) (10 points)
If you were unable to make the approximations $\phi \ll 1$ and $\theta \ll 1$, you would need to solve this system numerically. This can be easier if you use a set of first-order coupled partial differential equations, rather than a set of second-order partial differential equations. Given the Lagrangian, how would you obtain this first-order set of equations? (Only an explanation of what you would do is required.)
Solution:
The desired set of first-order coupled partial differential equations are Hamilton's equations

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi}$$
$$\dot{p}_\phi = -\frac{\partial \mathcal{H}}{\partial \phi}$$
$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial p_\theta}$$
$$\dot{p}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} \ .$$

The canonical momenta are

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$
$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \ .$$

The Hamiltonian is

$$\mathcal{H} = \dot{\phi} p_\phi + \dot{\theta} p_\theta - \mathcal{L} \ ,$$

where $\mathcal{H}$ must be reexpressed in terms of ϕ, θ, p_ϕ, and p_θ.
(**d**) (10 points)
Again for the conditions of part (**c**) (ϕ and θ not necessarily small), assume that you have an ideal double pendulum, an arbitrarily fast and precise computer, and "fairly accurate" initial conditions. Would you expect to obtain a "fairly accurate" prediction for its motion? Would your expecations depend on the range of motion that is considered? Explain.
Solution:
When the double pendulum with equal bob masses was simulated numerically by Hand, he found that the difference between solutions for infinitesimally different initial conditions grew exponentially with the number of periods considered, when the initial angle of the top bob was large (90°). Both Hand's pendulum and the present pendulum have irrational winding numbers, so we assume that the behavior of the current pendulum would be similar: when the initial angles are sufficiently large, "fairly accurate" predictions can't be made, even if the initial conditions are known with fair accuracy.

5. (40 points)
A physical system is described by a single dimensionless generalized coordinate $b(s,t)$ that is a function of two independent variables: a time variable t and a (one-dimensional) field variable s. When s and t are normalized (scaled) to be dimensionless, and the Lagrangian density $\mathcal{L}'$ is similarly renormalized, $\mathcal{L}'$ takes the form

$$\mathcal{L}'(b, \tfrac{\partial b}{\partial s}, \tfrac{\partial b}{\partial t}, s, t) = \tfrac{1}{2}\left(\tfrac{\partial b}{\partial s}\right)^2 - \tfrac{1}{2}\left(\tfrac{\partial b}{\partial t}\right)^2 \ .$$

(**a**) (10 points)
Using the version of the Euler-Lagrange equation that is appropriate for a Lagrangian density, show that the equation controlling the evolution of $b(s,t)$ is

$$\tfrac{\partial^2 b}{\partial s^2} - \tfrac{\partial^2 b}{\partial t^2} = 0 \ .$$

You may use this result in the remainder of this problem.
Solution:

$$\frac{d}{ds}\frac{\partial \mathcal{L}'}{\partial \frac{\partial b}{\partial s}} + \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \frac{\partial b}{\partial t}} = \frac{\partial \mathcal{L}'}{\partial b}$$
$$\tfrac{\partial^2 b}{\partial s^2} - \tfrac{\partial^2 b}{\partial t^2} = 0 \ .$$

This is a wave equation with unit phase velocity.
(**b**) (10 points)
If $-\infty < s < \infty$, *i.e.* there are no boundaries for s, what is the *general* solution $b(s,t)$ to this equation?
Solution:

$$b(s,t) = b_+(s-t) + b_-(s+t) \ ,$$

where b_+ and b_- are any two differentiable functions of their arguments. This describes a shape

b_+ propagating in the $+s$ direction, and a shape b_- propagating in the $-s$ direction, each with phase velocity equal (in these coordinates) to unity.

(**c**) (10 points)

Now impose the boundary condition

$$b(s=0,t) = b(s=1,t) = 0 \ .$$

What are the angular frequencies of the normal modes of this system?

Solution:

When boundary conditions are imposed, the solutions become sinusoidal standing waves of the form

$$b(s,t) \propto \sin ks \cos kt \ ,$$

where the harmonic functions are chosen to satisfy the particular boundary conditions that are imposed. Here we choose $\sin ks$ because $b(s=0)=0$. The boundary condition $b(s=1)=0$ is satisfied for $k=\pi,\ 2\pi,\ 3\pi...$ Therefore the normal angular frequencies are $\omega=\pi,\ 2\pi,\ 3\pi...$

(**d**) (10 points)

Finally, retaining the boundary condition introduced in part (**c**), impose the initial conditions

$$b(s,t=0) = \sin \pi s$$
$$\tfrac{\partial b}{\partial t}(s,t=0) = 0 \ .$$

What is the earliest time t_0 such that

$$b(s,t_0) = -b(s,t=0) \ ,$$

i.e. the field $b(s,t)$ reverses sign but is otherwise unchanged? Explain your reasoning.

Solution:

The boundary condition is such that only the first Fourier component k_1 is excited. At $k_1 t_0 = \pi$, it will change sign. Therefore, since $k_1 = \pi$, $t_0 = 1$.

Alternatively, you may use the fact that half of the initial waveform propagates to the left and half to the right. Each is inverted at the boundary. After a total propagation distance of $\frac{1}{2}+\frac{1}{2}=1$, the two inverted waveforms recombine. Again the elapsed time is $t_0 = 1$.

6. (25 points)

A one-dimensional physical system with generalized coordinate q and canonically conjugate momentum p is described by a Hamiltonian $\mathcal{H}(q,p,t)$ that is a smooth function of the variables upon which it depends. This is a conservative system (no dissipation), so that $d\mathcal{H}/dt$ vanishes, *i.e.* $\mathcal{H}$ is a constant of the motion.

(**a**) (10 points)

Prove that $\partial\mathcal{H}/\partial t$ vanishes.

Solution:

$$\begin{aligned}
\frac{d\mathcal{H}}{dt} &= \frac{\partial\mathcal{H}}{\partial t} + \frac{\partial\mathcal{H}}{\partial q}\dot{q} + \frac{\partial\mathcal{H}}{\partial p}\dot{p} \\
0 &= \frac{\partial\mathcal{H}}{\partial t} - \dot{p}\dot{q} + \dot{q}\dot{p} \\
0 &= \frac{\partial\mathcal{H}}{\partial t} \ ,
\end{aligned}$$

where Hamilton's equations are used in the next to last line.

(**b**) (15 points)

This system also is characterized by a different smooth function $F(q,p,t)$ of the same variables. It is known that F is also a constant of the motion. The quantity $\partial F/\partial t$ describes the *explicit* time dependence of the function F; it can be nonzero even when F is conserved. Prove that $\partial F/\partial t$ is a constant of the motion.

Solution:

$$\begin{aligned}
0 = \frac{dF}{dt} &= \frac{\partial F}{\partial t} + [F,\mathcal{H}] \\
\frac{d}{dt}\frac{\partial F}{\partial t} &= \frac{d}{dt}[\mathcal{H},F] \\
\frac{d}{dt}\frac{\partial F}{\partial t} &= [\dot{\mathcal{H}},F] + [\mathcal{H},\dot{F}] \\
&= [0,F] + [\mathcal{H},0] \\
&= 0 \ .
\end{aligned}$$

Analytic Mechanics: Discussion Worksheet 1

Robin Blume-Kohout

September 7, 2000

This worksheet deals with some of the mathematics essential for your understanding of mechanics. We'll work, later on, with techniques of calculus and geometry, but for now we're going to introduce some ideas of linear algebra and some notation.

1 Vectors and matrices

Let the following variables represent matrices (some also represent vectors):

$$A = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & +\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix} \quad E = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \quad F = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 1 & \frac{2}{3} \\ 3 & \frac{3}{2} & 1 \end{pmatrix}$$

1.1 Which pairs of the preceding matrices may be added?

1.2 Which pairs of the preceding matrices may be multiplied?

1.3 Of all of the allowed multiplications in the previous question, calculate explicitly both the least complicated (smallest number of scalar multiplications) and most complicated (largest number of scalar multiplications).

2 Tensors and component notation

In this section, all the matrices from the previous section will be considered as tensors. There is a technical question about whether non-square matrices should properly be described as tensors, but this will not be an issue after this point.

2.1 Consider a vector $\vec{a}$. Write (symbolically) the ith component of $\vec{a}$. If $\vec{a}$ is a 3-vector of the form (x, y, z), how would we indicate y in component notation?

2.2 Consider the matrix F from the previous section as a tensor. We will denote the number in the ith row and the jth column as F_{ij}. Can you find a formula that gives F_{ij} for all appropriate i and j?

2.3 If each matrix in the previous section is considered as a tensor, what is the rank of each?

a) b) c) d) e) f)

2.4 If the matrices in the previous section are considered as tensors, which are "symmetric" and which are "antisymmetric?" For which does the question even make sense?

a) b) c)

d) e) f)

2.5 The Levi-Civita tensor ϵ is described as "the totally antisymmetric unit tensor of rank 3." Given this definition, the additional (technically unnecessary) hint that its indices range from 1 to 3, and the (necessary) convention that $\epsilon_{123} = 1$, can you write down all the components of ϵ?

3 Tensor contraction, and thence to matrix multiplication

3.1 **Given the matrices A and B below,calculate $C = A \cdot B$ and $D = B \cdot A$. Now, write a component-notation formula for C_{ij} (feel free to use subscripts like k, l, m, n, a, b, c, etc.) and D_{ij}. Check a few components of C and D to make sure that your formula works (don't forget to sum!) If we let $E = C \cdot A + D \cdot B$, what is E_{ij} in terms of components of A and B?**

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 0 & +1 \\ -1 & 2 \end{pmatrix}$$

$C =$ $\qquad\qquad$ $D =$

$C_{ij} =$ $\qquad\qquad$ $D_{ij} =$

$E_{ij} =$

3.2 **P, Q, and R are square 2-tensors (from now on, all tensors are square, cubic, hypercubic, etc.) of the same order (dimensionality, or size). P and Q are symmetric, R is antisymmetric. Classify the following as symmetric, antisymmetric, or neither. Justify your reasoning (i.e., show your work).**

3.2.1 $X : X_{ij} = P_{ik}Q_{kj} + Q_{ik}P_{kj}$

3.2.2 $Y : Y_{ij} = P_{ik}Q_{kj} - Q_{ik}P_{kj}$

3.2.3 $Z : Z_{ij} = Q_{ik}R_{kj} - R_{ik}Q_{kj}$

3.2.4 $T : T_{ij} = P_{ik}Q_{kl}R_{lm}Q_{mn}P_{nj}$

3.3 Let u and v be 1-tensors. Let S and T be 2-tensors. What sort of entity is each of the following?

3.3.1 $u_i S_{ij}$

3.3.2 $T_{ij} u_i$

3.3.3 $u_i v_j$

3.3.4 $u_i v_i$

3.3.5 $T_{ij} S_{ji}$

3.3.6 $u_i T_{ij} S_{jk} v_k$

3.3.7 $u_i T_{ij} v_j$

3.3.8 $u_i T_{jk} S_{lm} v_k$

4 Give several (3 or more) definitions of what a tensor is. Strive for intuition and clarity, not precision – this isn't a graded exercise!

Analytic Mechanics: Discussion Worksheet 2

Robin Blume-Kohout

September 6, 2000

This worksheet is primarily concerned with transformations and rotations. The point today is to become so doggone comfortable with the philosophy, mathematics, and conventions of rotations (and transformations in general) that you never have to think about it again. We'll fail, of course, but we'll give it a good try!

1 Define, in a sentence or two or three, the following terms. Try to be *insightful* and *instructive* as opposed to *precise*; this is for understanding, not for a grade.

1.1 Rotation

1.2 Space point

1.3 Space axes

1.4 Body axes

1.5 Passive rotation

1.6 Active rotation

1.7 Line integral

1.8 Conservative force

1.9 Field (physics, not math)

1.10 Force field

1.11 Pseudovector (and why does it behave that way?)

2 Rotation Conventions

I'm in a spaceship near earth, and I want a rotation matrix to describe the earth's rotational motion over a period of 6 hours (i.e., the discrete transformation from noon to 6 PM, not every time in between). Let $\hat{e_3}$ point from the south pole to the north pole, and let $\hat{e_2}$ point from the center of the earth to the sun.

2.1 Find the rotation matrix $\hat{\Lambda}$.

2.2 Is this an active or passive rotation?

2.3 Now, let $\hat{e_3}$ point in a direction halfway between the pole-pole line and the center-sun line. Find $\hat{\Lambda}$ either explicitly or as a product of matrices.

3 Is there any reason why I couldn't define the origin of the body frame to have the space coordinates $\vec{O'} = (1, 0, 0)$? Why is this acceptable, or why is it not?

4 Give 3 rotation matrices that will move the north pole of a sphere to its south pole. Would this be an active or passive rotation, in your opinion? Note: at least one of your matrices should have a different determinant from the others.

5 Euler-like Rotations

You are looking at a unit sphere in space. You want to rotate your point of view (what kind of rotation is this?) so that the point on the sphere at $(0, \sqrt{2}, \sqrt{2})$ goes to $(\sqrt{2}, \sqrt{2}, 0)$ and $(0, 0, 1)$ goes to $(0, 1, 0)$.

5.1 Draw a picture of this rotation using your best 3D drawing skills.

5.2 Describe two ways of building this rotation R out of 3 simple rotations, one of which should be according to the Euler convention.

5.3 Calculate the rotation matrix $\hat{R}$ both ways, and ensure that they agree.

5.4 Are any points left unmoved by this transformation? Why or why not? If you choose "yes," describe them.

5.5 How would you calculate such points if they existed?

6 Hyperbolic transformations (optional)

The exponential of a matrix, $e^{\hat{M}}$, is defined by the power series of the exponential. This includes only powers of the argument to the exponential, and powers of a matrix are well defined.

6.1 Show that all rotations in 2 dimensions can be written in the following form:

$$\begin{aligned}\hat{R} &= e^{\theta\hat{\sigma}} \\ \hat{\sigma} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\end{aligned}$$

6.2 Now, consider another transformation just like the previous one, but where the exponentiated matrix (the "generator" of the transformations) is given by:

$$\hat{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In this case, what is the form of the transformation matrix?

6.3 Is the norm of a vector $\hat{r}$ preserved? (That is, $\hat{r}\cdot\hat{r}$.) What quantity *is* preserved?

6.4 Does your answer to the previous question give you a hint as to how to combine two vectors to form an *invariant* quantity – that is, how to redefine the dot product to be invariant under this transformation rule?

Analytic Mechanics: Discussion Worksheet 3b

Robin Blume-Kohout

September 14, 2000

Consider a scalar function of two variables, $f(x, y)$. The partial derivative of a function f with respect to a variable x, $\frac{\partial f}{\partial x}$, is calculated by assuming that *all other variables* on which f depends are constants. This yields a new function which we'll call $f|_y(x)$ (read "f at constant y of x") which is dependent on x. We take a plain old derivative of $f|_y$ with respect to x, and that's $\frac{\partial f}{\partial x}$.

1 Given $f(x, y) = sin(x)sin(y)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

2 Let's define $p = \frac{1}{\sqrt{2}}(x + y)$ and $q = \frac{1}{\sqrt{2}}(x - y)$.

2.1 On an $x-y$ plot, sketch the lines $p = 0$ and $q = 0$. Why do we assert that p and q are "independent coordinates?"

2.2 Write the f given previously in terms of p and q. That is, find a function $f_{new}(p, q)$ such that $f_{new}(p, q) = f(x, y)$ if p and q are defined as above.

2.3 Calculate $\frac{\partial f}{\partial p}$ and $\frac{\partial f}{\partial q}$ explicitly. How could you write expressions for these in terms of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$?

2.4 Suppose you have a coordinate s given by $s = \hat{n} \cdot \vec{r}$, $\vec{r} =< x, y >$. How would you calculate $\frac{\partial f}{\partial s}$, given $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$?

3 What is the difference between the following expressions?

$$Q_1 = f(x_0, y_0) + \int_{x_0}^{x_f} \frac{\partial f}{\partial x} dx \quad (1)$$

$$Q_2 = f(x_0, y_0) + \int_{x_0}^{x_f} \frac{df}{dx} dx \quad (2)$$

3.1 Which of the two is "incomplete" – that is, requires the definition of another quantity to make it meaningful (assuming that f, x_0, and y_0 are all defined)?

3.2 Modify one of the expressions to make it identical to the other.

4 Define the vector $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \ldots\right)$, where for any number of dimensions, $\nabla_i \equiv \partial_i \equiv \frac{\partial}{\partial x_i}$. Show that $\vec{\nabla}$ is a vector (in the physics sense).

5 Invent at least four ways to combine the $\vec{\nabla}$ vector operator with one sort of field or another to construct scalar, vector, or tensor fields.

6 Suppose you have a vector field $\vec{F}(\vec{r})$. Now we perform a rotation $\hat{R}$ on the field and end up with a new field $\vec{F}'$. Give an expression for $\vec{F}'(\vec{r}')$.

Analytic Mechanics: Discussion Worksheet 4

Robin Blume-Kohout

September 21, 2000

This week, we're going to work on some very basic Lagrangian mechanics problems, with an eye toward slowing down (compared to the torrid pace of the homeworks) and trying to understand every step of the process.

1 Ruminations on the free particle

1.1 Write down the Lagrangian for a free particle moving in two dimensions, x and y.

1.2 Use this Lagrangian to calculate the canonical momenta that are conjugate to x and y, p_x and p_y respectively.

1.3 Suppose that you added a potential $U(x, y)$ to your system. Write out the Euler-Lagrange equations, using p_x instead of $\frac{\partial \mathcal{L}}{\partial \dot{x}}$ (and similarly for y). Do you recognize these equations from 7A?

1.4 Forgetting about $U(x, y)$ now, assume that the particle is compelled to move in a circular path of diameter 1, centered at the origin.

1.4.1 Write the equation of constraint.

1.4.2 Identify g_x and g_y for use with Lagrange multipliers.

1.4.3 Write out the full Euler-Lagrange equations, with undetermined multipliers.

1.4.4 **Differentiate the equation of constraint twice, and solve for the acceleration in the x-direction.**

1.4.5 **Use what you've written so far, along with conservation of energy, to solve for the force of constraint required to keep the particle in a circular trajectory. Having found this, can you write it in a nice vectorized form?**

1.4.6 **[optional] The forces of constraint can be integrated to find an effective Lagrangian that would have yielded the solution to the constrained problem *without* any constraints. Try to write down such an effective Lagrangian, either explicitly or in integral notation.**

1.4.7 **[optional] In the previous section, can you write the Lagrangian in terms of coordinates such that one is cyclic? (Think; it's a trick question.) If so, what is the cyclic coordinate? If not, why not?**

2 Particle in a gravitational field

2.1 Again in two coordinates, write the Lagrangian for a particle in a uniform gravitational field $U = mgy$. Solve for the motion of the particle.

2.2 Suppose that the particle is constrained to a parabolic surface, $y = -x^2$.

2.2.1 **Write the equation of constraint, and find g_x and g_y. Are these choices of g's unique? Why [not]?**

2.2.2 **Write out the full (including multipliers) Euler-Lagrange equations.**

2.2.3 **Use the E-L equations with the equation of constraint to find an equation of motion for the system.**

2.2.4 **Find the point at which the force on the parabolic constraint changes sign (direction).**

Homework #5: Hints

Robin Blume-Kohout

September 29, 2000

1 Particle in an EM field

Write down the equation of motion, using complex notation (i.e., replace $\cos\omega t$ with $e^{i\omega t}$). Solve by guessing the form of the solution and plugging into the equation. Note that you are asked to find the *steady state* solution – this means the particular solution, not the homogeneous solution. That's why the amplitude and phase are not adjustable, and why you can determine them from the parameters given.

2 The Damped, Undriven oscillator

(a)

Write down (derive or copy from the notes) the general solution to the undriven, damped oscillator. Calculate $E(t) = T(t) + U(t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2x^2$. Find t' such that $E(t') = \frac{1}{e}E(0)$, and calculate N, the number of periods, from t'.

(b)

First, calculate the energy stored in the oscillator by solving the particular equation (use the method we used in (1) of assuming the form of the solution with adjustable parameters for the amplitude and phase, and plugging into the equation to find them). You can assume a driving force of the form $F = F_0\cos\omega t = \Re(F_0e^{i\omega t})$. Next, calculate the work done by the driving force in one period by calculating:

$$W = \oint F dx = \int_{t_0}^{t_0+2\pi\omega} F(t)v(t)dt$$

You should know $v(t)$ by differentiating $x(t)$ that you found above.

3 Specific Solutions to a Damped Oscillator

Both parts are solved almost identically. Start by writing down the general solution to the undriven oscillator, either underdamped or overdamped (this is the part that's different between the two solutions. Find the solutions in the notes if you can't rederive them; you should feel free to use anything given in the lecture notes unless specifically told to rederive it). Now, just plug in the values that you're given for Q to find either ω_γ or γ in terms of ω_0. You should have two adjustable constants left; use the boundary conditions given in the problem to fix those, and write out the final solution with no parameters except x_0, ω_0, and t.

4 The Critically Damped Oscillator

Start, as usual, by writing out the general (particular) equation for a damped, driven oscillator, in complex form. Assume the same form of the solution that we always do, and solve for the amplitude and phase of the particular solution $x_p(t)$. Now, write out the homogeneous equation, and either solve it for $Q = \frac{1}{2}$ or use the overdamped solution with the substitution $\gamma = 2\omega_0$. Add this solution ($x_h(t)$) and the particular solution, and fix the constants using boundary conditions. $x(t)$ should have no constants in it except G and ω_0. NOTE: you *are* given $\dot{x}(0)$ in the problem, if you think about it.

5 Woofer Design

(a)

What happens if you displace an overdamped oscillator? What about if it's an underdamped oscillator? Does this lead you to an answer?

(b)

Write down the given equation of motion, and divide through by m to get it in familiar form. Identify γ and ω_0. You are given ν_0; use that to find ω_0. That, together with Q, should give you a value for γ.

As usual, find the amplitude as a function of various parameters using the particular equation. The key insight is that the sound intensity is proportional to the square of the sound *amplitude* – and the sound amplitude is proportional to the force exerted by the cone on the air around it. What is the force exerted by the cone on the air proportional to? That should give you the sound intensity (to within a constant factor) in terms of ω and A. Now, just calculate the ratio $I(\omega)/I(\omega_0)$ for the given frequencies.

(c)

Basically, what you should do is try to determine how ω_0 and γ depend on the cone area A. To find this, consider how m,k, and b depend on A. You should find that one of $\{\omega_0, \gamma\}$ is independent and one is proportional to A^{-1}. Then define $z = \frac{A}{A_{\text{initial}}}$, plug the z-dependence of ω_0 and γ into the expression for intensity from the previous section, and plot it for various values of z. At what values of z does the curve that you get look "smooth" – that is, has no spikes or small spikes?

6 Fourier Series

First of all, you must assume that the function is periodic, although the definition given in the problem doesn't make it look that way. That is, $F\left(t + \frac{4\pi}{\omega}\right) = F(t)$.

Next, find the period of the function. It's basically given to you in the hint above. Now, write out equation (3.12) from the reader, and use equation (3.13) to find the coefficients f_n and g_n. You're on your own doing the integrals; I recommend looking them up or using a computer program like Maple or Mathematica. Once you have the coefficients, just write out equation (3.12) with the coefficients in place, and you're done.

7 The Funkily Driven Oscillator

Keep in mind that you need only the particular solution, not the homogenous solution (transients have been damped out).

Start by writing out the particular equation, with the whole nasty force term on the right (in the Fourier form given in problem [6]). You should have a driving term:

$$\frac{F_0}{2m}\sin\omega t + \sum_{n \text{ odd}} \frac{4}{\pi(4-n^2)}\frac{F_0}{m}\cos\frac{n}{2}\omega t$$

Use that form if you didn't get the right answer for (6), or go back and fix it in (6). As I explained in discussion section (for those of you who were there), the solution to a sum of driving terms is simply the sum of the solutions to the individual driving terms, so you can write the solution as $x(t) =$ the real part of a sum of complex exponentials, where the frequency of each complex exponential corresponds to the frequency of a driving term given in the expression above. You should have one term corresponding to the first $(\sin \omega t)$ term above, and a sum of terms indexed by n corresponding to the sum of driving terms above. Each term in the solution should have its own independent amplitude, perhaps A and B_n.

Now, solve the equation separately for each ω in the sum. Remember to do this all in the complex exponential formulation, and write your final answer as the real part of the complex solution that you get. The final answer is heinously long and nasty; please write it neatly so that I can check to make sure it's right or correct it!

8 Overdamped Green's Function

The Green's Function is simply the solution to the particular equation with a δ function driving term. That is, it's the solution to the equation:

$$\ddot{X}_g(t) + \gamma \dot{X}_g(t) + \omega_0^2 X_g(t) = \delta(t)$$

For those of you who don't feel comfortable with delta functions, $\delta(t)$ spikes at $t = 0$ and is zero everywhere else; it's infinite at the origin, but in such a way that $\int_{-\infty}^{\infty} \delta(t) dt \equiv 1$.

For all $t > 0$, the driving term is zero, so the particular equation becomes the homogeneous equation there. Why don't you have to worry about $t < 0$?

When you solve the homogeneous equation for $t > 0$, you find there are undetermined constants. You should solve for those using the initial conditions $X_g(0)$ and $\dot{X}_g(0)$ – keeping in mind that you really want to use the values of X_g and its derivative just *after* $t = 0$ – that is, after the delta function drive occurs. Use the concept of *impulse* to find the velocity immediately after the "kick" is applied at $t = 0$. Once you've matched these conditions, you're done.

HW #6: Hints

Robin Blume-Kohout

October 4, 2000

1 Period of Oscillation in a Power-Law Potential

First of all, since the problem only asks you to find the period of oscillation, you don't need to find $x(t)$. In fact, you can't. Just use energy conservation to calculate x_{max} and x_{min} for the particle, and to find $\dot{x}(x)$. Then use the fact that $dt = \frac{dt}{dx}$ (how can you find $\frac{dt}{dx}$?) to set up an integral for part of the period (it's easier to do a fraction of the whole period than the whole thing at once; just calculate the total period from this particular fraction.)

You'll have to do a substitution to get the integral into the following form:

$$\int_0^1 \frac{y^{\frac{1}{n}-1}dy}{\sqrt{1-y}}$$

Of course, everyone learned that integral in junior high school. Not. It's a β function, with the property

$$\beta(p,q) \equiv \int_0^1 t^{p-1}\,(1-t)^{q-1}\,dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

If you don't know what the Γ function is, it's much like the factorial function – specifically, $\Gamma(n+1)=n!$ for all integer $n > 0$. However, the Γ function is also defined at non-integer n; it's defined as:

$$\Gamma(x) \equiv \int_0^\infty t^{x-1}e^{-t}dt$$

Who would have thought that would give you a factorial? Anyway, you don't need to know all that for the problem, but it's good to know about Gamma functions. The only thing you need to know is that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

2 Underdamped Oscillator

You should find the Green's function for the underdamped oscillator in the lecture notes. Recall that because the oscillator is linear,

$$D \sum_i x_i(t) = \sum_i \frac{F_i}{m}(t)$$

if $x_i(t)$ is the solution to the driving term $\frac{F_i}{m}(t)$. Thus, if you have the solution to a delta function driving term:

$$D\left[G(t-t')\right] = \delta(t-t')$$

then by writing an arbitrary driving term as $F(t) = \int_{-\infty}^{\infty} F(t')\delta(t-t')dt'$, we can write the solution as

$$x(t) = \int_{-\infty}^{t} F(t')G(t-t')dt'$$

The reason for not letting the upper limit of integration go to ∞ is to deftly include the requirement that $G(t) = 0$ for $t < 0$. That makes sense, right? – it just means that a driving force that *happens* at $t = 0$ cannot cause motion at $t < 0$. Alternatively, if t' is later than t, then a driving force at t' can't cause motion at t. It's easier to change the limit of integration than to explicitly make $G(t < 0) = 0$ – look in the notes and you'll see that $G(t-t')$ has a nice functional form as long as you *implicitly* add the assumption that $G(t < 0) = 0$.

Finally, once you've set up the integral, I recommend you do a slight substitution to set the origin of time $t = 0$ to a convenient locale. From there it's just math (though the answer is rather heinous). Once you get an answer, you may wish to check the answer in the solution set, and fix any mistakes you find (better than me correcting them when I grade, 'cause this way you get your feedback immediately!)

3 Perturbation Theory for a Nonlinear Oscillator

Your strategy is as follows (this is the fundamental method of "perturbation theory" in any branch of physics). Assume that λ is a small number, so that λ^2 is *much* smaller than λ. Then, since you don't know the solution, you can quite generally write it as a *power series* in λ – that is,

$$x(t) = x_0(t) + \lambda x_1(t) + \lambda^2 x_2(t) + \ldots$$

but you immediately decide to ignore all terms with more than one power of λ throughout the solution (that's what's meant by "accurate to first order"), so $x(t) = x_0(t) + \lambda x_1(t)$.

Now plug it into the differential equation, expand things out, and ditch all the λ^2 and λ^3 terms (etc, etc). Once you've got your equation to 1st order in λ, you can separate into two equations, one of which has terms with no powers of λ and one of which has terms proportional to λ, on the philosophy that this has to be valid no matter what the value of λ is. Think of it as similar to taking a complex equation with real and imaginary parts and separately solving for the real and imaginary parts.

Anyway, one of those equations will give you a solution for $x_0(t)$. (Don't forget to match initial conditions!) You can take that solution and plug it into the other equation. You get another equation that's familiar, but with a weird driving term. Specifically, it has an absolute value in it! Just consider separately the first half-period where the term in $|f(t)|$ is equal to $f(t)$ and the second half-period where $|f(t)|$ is equal to $-f(t)$.

You may want to guess a particular solution of the form $x_{1p}(t) = A + B\cos 2\omega_0 t$. Then add the homogenous solution $x_{1h}(t)$ and match boundary conditions. That should give you $x_1(t)$ for the first half-period... now do it for the *other* half-period when the driving force changes because of the absolute value.

When matching boundary conditions, use the fact that x and $\dot{x}$ must be continuous if there is no infinite force!

Finally, when you've done all this for the 2nd half-period, see what you end up with at $t = 2\pi\omega_0$. See how this frees you from solving the problem over and over again for each and every succeeding half-period. *Hint (sub-hint?)*: the oscillator is damped.

4 Particles w/Gravitational Force

First of all, as always occurs with central-force problems, you'll want to view this as an effective one-body problem – r is the distance between an "effective particle" of mass $\mu = \frac{m_1 m_2}{m_1+m_2}$. You should be able to find the period of the orbit in terms of the radius R of the orbit, using a simple 7A argument about uniform circular motion.

If you're confused by the statement of the problem, then rest assured that when the particles are "stopped in their orbits," they are instantaneously placed *at rest*: $v = 0$. So you can use energy conservation to find $\dot{r}(r)$ as in problem (1), and use a very similar integration technique (remember the

β function?) to find $T_{collision}$. To obtain the final answer, use the values of the Γ function that I gave you in problem (1), together with the fact (generalized from the factorial function) that

$$\frac{\Gamma(x+1)}{\Gamma(x)} = x$$

5 Space Explosions!

This is a pretty straightforward problem – I don't think you'll need much help. However, you may want to consider the implications of the virial theorem (see notes) for the relative kinetic energy of a nose cone in circular and parabolic orbits. What does this imply for the change in momentum of the nose cone? and therefore for the change in momentum of the service module, which (I remind you) falls "directly," or straight, into the sun. Use conservation of momentum to find the relative masses.

6 Central Power Law Potential

First of all, try playing around with the equations that you have in order to write the total energy of the particle as a function of r only. Specifically, you gotta write $\dot{r}$ and $\dot{\theta}$ in terms of r. If you can't get it, here's a more specific hint.

Try writing the orbit (a circle that passes through the origin) in polar coordinates. Don't worry about time, just parametrize the curve as $r(\theta)$. Remember that everything that is usually conserved ($\vec{p}$, $\vec{L}$, and E) is conserved, and use the conservation laws together with the time derivative of the orbit equation (i.e., $r(\theta)$) to write down the same sort of $\dot{r}(r)$ relation we've already used twice.

Now, you can write the kinetic energy in terms of r and you can also write the potential energy in terms of r (you're given F, right? and $F = -\frac{\partial U}{\partial r}$, right?). So just make sure that the total energy is (because it's conserved) a constant that doesn't depend on r – which means that any powers of r that you have in E must (a) be the same, and (b) cancel each other out term by term.

7 Spaceship Jitters

Note that only the direction of the velocity vector changes when the engines fire.

How does E_f, the energy after firing, compare to E_i, the energy before? What about L_f versus L_i (hint: write the new velocity in polar-coordinate vector form)?

Use (7.12) and (7.10) from the notes to use your information about the change in L and E to find the new eccentricity.

8 Boing, Boing

(a)

Set this up as a Lagrangian problem, using generalized coordinates based on the position of the puck on the table (that determines the height of the weight). What's conserved here (hint, look at the Lagrangian, and the E-L equations if necessary)?

When you calculate the E-L equation for r, you'll get an equation of motion for $\ddot{r}$. This gets even simpler for a circular orbit – think about the behavior of r for a circular orbit. The trick here is to ignore θ and imagine perturbing a circular orbit of radius R by a small amount ϵ – that is, $r = R + \epsilon$. Use the E-L equation to find the equation of motion for ϵ, and simplify it by a Taylor expansion of the nonlinear term for $\epsilon << R$. Since we're talking oscillations, you had better end up with an equation for ϵ like $\ddot{\epsilon} \propto -\epsilon$. That gives you the frequency of perturbation.

(b)

If the ratio of two frequencies is irrational, then the orbit can never close. If you use this hint, I'd like to see a creative (well, some kind of) explanation for *why* this is true – not just a statement that it is true. Oh... and you should certainly show that my statement applies to this case!

Notes on Hamilton's Principle

Robin Blume-Kohout

October 2, 2000

1 Introduction

In class, we mentioned that the justification for Lagrangian mechanics is something called Hamilton's Principle. Usually, Hamilton's principle is known as the "principle of least action," and it's assumed to say the following: *The path followed by a system in transitioning from a point* $(\vec{x_1}, t_1)$ *to a point* $(\vec{x_2}, t_2)$ *is the path which minimizes the action,* $S[x(t)] = \int_{t_1}^{t_2} \mathcal{L} dt$. In these notes, I will briefly explain why this is incorrect, what the correct statement is, and why (physically) the world works that way. In addition, I'll explain how we know the second end-point of the trajectory – that is, how the system decides "*where* to end up" so that we can calculate the path it takes to get there.

2 Hamilton's Principle

Historically speaking, Hamilton formulated his eponymous principle *after* Lagrangian mechanics had already been conceived of. It is possible to derive Lagrangian mechanics directly from Newtonian mechanics, but formulated in that way it contains no new insights on how the universe works. Hamilton noticed that the Lagrangian formulation could also be derived by assuming that that all physical paths are minimum-action paths, and therefore proposed the axiomatic principle of least action (that is, it's axiomatic because it can't be proved, and must be assumed).

You may recall that the way Lagrangian mechanics works is by extremizing the action along a path by setting a sort of derivative $\frac{\delta S}{\delta \epsilon}$ (the derivative of the action with respect to some parameter of the path) to zero, in analogy with the technique for finding the minimum of a function in calculus. We mentioned that this technique, as in calculus, only guarantees that we have found *one* of three possibilities: a minimum, a maximum, or a point of inflection (if you don't remember what a point of inflection is, then consider that $f(x) = x^3$ has one at $x = 0$; it's neither a min nor a max, but $\frac{df}{dx} = 0$.) Generally, it's assumed that if we find a path where $\frac{\delta S}{\delta \epsilon} = 0$, then according to Hamilton's principle it's a minimum.

In reality, this is not true. Frequently, the path found by applying the Euler-Lagrange equations to the action of a physical system is either a point of inflection or a local maximum! This immediately begs the question: why does the method work? The answer is simple; Hamilton's principle as stated previously is too restrictive. It should be restated as the *principle of stationary action*, saying that the path that a system follows is one where the action is *stationary* with respect to variation in the path – that is, precisely that $\frac{\delta S}{\delta \epsilon} = 0$! Thus, the Lagrangian method always works.

Why is this the case? The principle of least action is, at least, intuitively appealing – it's elegant and satisfying that nature should choose to minimize some quantity, even if we don't know why. This idea of stationary action is much less satisfying – it seems almost unfairly simple that just when it turns out that our method for finding minima is flawed because it also finds maxima and points of inflection, it also turns out that nature will accept either of the others as well! The answer comes from quantum mechanics.

3 A Brief Justification for Stationary Action

Since this is not a class in quantum mechanics, all I'll attempt to do is give a brief rationale for why the paths with stationary action are the ones that the system follows. There are several perspectives on quantum mechanics, each of which yields the same results. This is analogous to the way in which we can derive the same results for a classical system using Newton's laws, Lagrangian techniques, or Hamilton's equations of motion.

One of these perspectives is called the *path-integral approach*. Using this approach, we can calculate the *probability* for a system, in the state corresponding to $x = x_1$ at time $t = t_1$, to be in the state $x = x_2$ at time $t = t_2$. The way we do this is as follows:

* Form a set $\mathcal{P}$ containing every conceivable path $x(t)$ that connects (x_1, t_1) to (x_2, t_2).
  For every path $x(t) \in \mathcal{P}$, calculate the action $S[x(t)]$. Notice that we've mapped each path (a function) to an action (a real number).

* Define the *amplitude functional* $A[x(t)]$ to be the complex exponential of the action: $A[x(t)] = e^{i\hbar^{-1}S[x(t)]}$.

* Now, add up $A[x(t)]$ for *every* $x(t) \in \mathcal{P}$, to get $A_{total}(x_2, t_2; x_1, t_1) = \int_{\text{all paths } x(t)} e^{i\hbar^{-1}S[x(t)]}$.

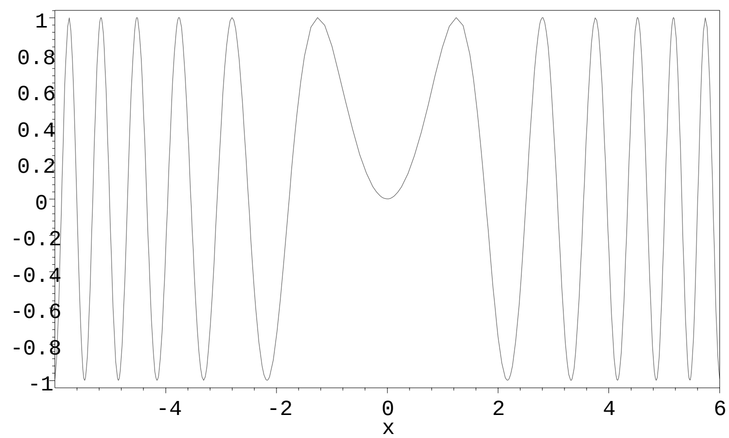
* Finally, the probability for the system to move from x_1 to x_2 is just $|A_{total}|^2$, the norm-squared of the amplitude.

Now, how does this esoteric procedure yield the principle of stationary action? Well, consider the behavior of the function $f(S) = e^{iS}$. As S increases, $f(S)$ oscillates around the unit circle in the complex plane. If you're not familiar with complex functions, just think of $f(S) = sin(S)$; it's pretty much the same for our purposes, except that since $sin(S)$ is a real function, it oscillates up and down in the real numbers. Now, what happens when we integrate this oscillating function of the action over all paths?

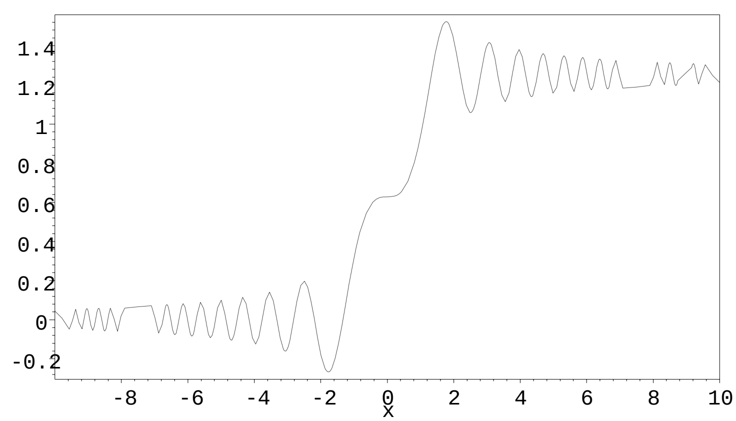
Well, the graphs on the next page should illustrate it. Basically, wherever the action is changing, the integrand e^{iS} is oscillating, and the integral over that region practically vanishes. In the example on the next page, I'm parametrizing my "paths" by a variable called y (not to be confused with the $x(t)$ we were talking about before; this is more like ϵ, but it's easier to use y in Maple, where these plots were done), and the "action" is $S = y^2$. I've shown a graph of the imaginary part of the amplitude, $\Im e^{iS(y)} = sin(S(y))$, and a graph of its integral over an interval from $-\infty tox$. Notice that the integral pretty much just wiggles around without going anywhere *until* we get to around $x = 0$. Why $x = 0$?

Well, at $x = 0$, the derivative of S with respect to the parameter y is zero – which means that the action doesn't depend on the path locally. That is – the action is *stationary* at $y = 0$! Because the integrand isn't oscillating around any more, this path contributes the lion's share of the final integral.

What do we conclude? That the particle has a significant probability of going from x_1 to x_2 *if and only if* there exists at least one path (connecting x_1 to x_2) around which the action is stationary. It doesn't have to be minimal, it doesn't have to be extremal, merely stationary. In classical mechanics, we take certain limits ($\hbar \to 0$, for those of you who like quantum mechanics) so that the phrase "significant probability of" can be replaced by "*any* possibility of". That is, likelihoods become certainties. This is where we get the principle of stationary action from.



$$f(x) = \Im(e^{(ix^2)})$$



$$\int_{-\infty}^{x} \sin(y^2)\, dy$$

4 How the particle knows where to go

Finally, I want to address a more practical question. When we did the calculus of variations to find the minimum action path, we always assumed fixed endpoints x_1 and x_2. How do we know what endpoints to choose? That is, for a given physical system starting at point (x_1, t_1), how do we know at which x_2 it's going to be at time t_2 so as to find the stationary-action path between the two?

The answer is twofold. First of all, a particle will not move under any circumstances between two points that are not connected by a stationary-action path. That is, if by some chance there does not exist any path of minimum or stationary action connecting x_1 to x_2, then the particle will never end up at x_2. This rarely happens, but it is possible – a simple example is a system which contains a region of infinite potential (positive or negative). The system will never end up in that region, because every path connecting the starting point to a point x_2 inside a region of infinite potential has infinite action – which means that there is no minimal or maximal action path. In practice, this is rarely an issue.

Secondly (and more relevantly), there is a sense in which x_2 is *not* determined. That is, there is almost always a path of stationary action connecting any two points x_1 and x_2. However, we haven't specified the initial *velocity* yet! You must remember that all this path-action stuff is dealing with two points in *configuration space*; we aren't talking about phase space. The end points x_1, x_2 determine the path, but that path determines the velocity $\dot{x}(t)$. So, for instance, I could analyze the physics of a ball dropped in a gravitational field, and find a stationary-action path connecting the points $z(t_1) = 0$, $z(t_2) = 0$ – which would imply that the ball does not fall, but stays in place! However, in doing so I find that the path which connects those two points requires the initial velocity of the ball to be upward, which makes sense.

If, however, I insist that the intial velocity of the ball be zero – it's dropped from rest – then the only path going *anywhere* that I can find which is stationary-action is our old friend, $z(t) = -\frac{1}{2}gt^2$. Thus, fixing the initial position *and* velocity (or momentum) is equivalent to fixing the end points.

This leaves at least one question to which I am not sure I have an answer, so you may want to think about it. Why is it, if we are going to claim that knowing $x_{initial}$ and $\dot{x}_{initial}$ is equivalent to $x_{initial}$ and x_{final}, that the conversion from one set of information (initial position-velocity) to the other (initial-final position) is always defined and invertible? That is, why is it that I never fail to find an $[x_1, x_2]$ pair to satisfy a particular $[x_1, \dot{x}_1]$ pair? I don't know yet... maybe you can figure it out.

Cheers,
Robin